Multi-User Privacy: The Gray-Wyner System and Generalized Common Information

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Abstract—The problem of preserving privacy when a multivariate source is required to be revealed partially to multiple users is modeled as a Gray-Wyner source coding problem with \(K\) correlated sources at the encoder and \(K\) decoders in which the \(k^{th}\) decoder, \(k = 1, 2, \ldots, K\), losslessly reconstructs the \(k^{th}\) source via a common link of rate \(R_0\) and a private link of rate \(R_k\). The privacy requirement of keeping each decoder oblivious of all sources other than the one intended for it is introduced via an equivocation constraint \(E_k\) at decoder \(k\) such that the total equivocation summed over all decoders \(E \geq \Delta\). The set of achievable \((\{R_k\}_{k=1}^K, R_0, \Delta)\) rates-equivocation \((K + 2)\)-tuples is completely characterized. Using this characterization, two different definitions of common information are presented and are shown to be equivalent.

I. INTRODUCTION

Information sources often need to be made accessible to multiple legitimate users simultaneously. However, not all data from the source should be accessible to all users. For example, a computer retailer may need to share the annual revenue of all computers sold with all the vendors but share vendor-specific sale information only with a particular vendor. Similarly, a business consulting firm may share general data about a specific market with all clients associated with that market but share client-specific strategies with only that client. In both cases, one can view sharing the public (shared by all) information via a common link and the private information via a dedicated link. Maximizing the rate over the common link allows the information source (retailer/consulting firm) to interact with the literature, this common rate is defined as the common information.

The common information of two correlated random variables has been defined independently by Wyner [1] and Gács-Körner [2]. Wyner’s definition of common information as applied to the two-user Gray-Wyner system (without privacy constraints) is the minimum rate on the common link such that the total information shared across all three links (one common and two private) does not exceed the source entropy. On the other hand, the Gács-Körner common information is the maximal entropy of a random variable that two non-interacting terminals can agree upon when one terminal has access to \(X^n\) and the other to \(Y^n\) where \(X\) and \(Y\) are correlated random variables. For two correlated variables \(X\) and \(Y\), the Wyner common information \(C_W\), the Gács-Körner common information \(C_{GK}\), and the mutual information of the two variables are related as \(C_{GK} \leq I(X;Y) \leq C_W\). Recently, the authors in [3] have generalized Wyner’s definition of common information to \(K\) variables, henceforth referred to as \(B \left( X_1, X_2, \ldots, X_K \right) \) for \(K\) correlated variables. While the definition naturally generalizes the two variable common information, the resulting common information does not satisfy a non-increasing property with \(K\) as expected.

In this paper, we present two different definitions of common information: the first is the maximal rate on the common link for which the total equivocation is maximized, and the second is the maximal rate on the common link such that each user losslessly reconstructs its intended source at its entropy.

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We show that both definitions lead to the same formulation for common information $C(X_1, X_2, \ldots, X_K)$. We present many properties of $C(X_1, X_2, \ldots, X_K)$ and specifically show that $C(X_1, X_2, \ldots, X_K) \leq B(X_1, X_2, \ldots, X_K)$. To the best of our knowledge this is the first generalization of common information that preserves the non-increasing property and one whose form can be viewed as a natural generalization of the Gács-Körner common information to $K$ variables.

The paper is organized as follows. In Section II, we present the system model. In Section III, we present the rate-equivocation region, develop a formulation for common information in two different ways, and present key properties. In Section IV, we compare our formulation with the $K$-variable generalization of Wyner’s common information in [3] and illustrate with examples. We conclude in Section V.

II. SYSTEM MODEL

We consider the following source network. A centralized encoder observes $K$ discrete, memoryless correlated sources, $\{X_K^n\}_{k=1}^K$ and is interested in communicating source $X_k$ to decoder $k$ in a lossless manner. The resources available at the encoder comprise two types of noiseless rate-limited links. There are $K$ links of finite rate from the encoder to each of the $K$ decoders and there is a common link of finite rate to all decoders. Figure 1 shows the source broadcasting network in consideration.

An $(n, \{M_k\}_{k=1}^K, M_0)$ code for this model is defined by $(K + 1)$ encoding functions described as

$$(f_0: \mathcal{X}_1^n \times \cdots \times \mathcal{X}_K^n \rightarrow \{1, \ldots, M_0\}),$$

and $K$ decoding functions

$$(g_k: \{1, \ldots, M_0\} \times \{1, \ldots, M_k\} \rightarrow \mathcal{X}_k^n, \quad k = 1, \ldots, K),$$

and the total equivocation as $E = \sum_{k=1}^K E_k$.

Remark 1: Informally, $E_k$ captures the average uncertainty, and hence privacy achievable, about the remaining $(K - 1)$ unintended sources at decoder $k$.

An $\{(R_k)_{k=1}^K, R_0, \Delta\}$ rate-equivocation $(K + 2)$-tuple is achievable for the source network if there exists an $(n, \{M_k\}_{k=1}^K, M_0)$ code such that

$$M_0 \leq 2^n R_0,$$

$$M_k \leq 2^n R_k, \quad k = 1, \ldots, K, \quad (3)$$

$$P_{e,k} \leq \epsilon_k, \quad k = 1, \ldots, K, \quad (4)$$

$$E \geq \Delta - \epsilon. \quad (5)$$

We denote by $\mathcal{R}$ the region of all achievable $\{(R_k)_{k=1}^K, R_0, \Delta\}$ rate-equivocation $(K + 2)$-tuples.

III. MAIN CONTRIBUTIONS

A. Rate-Equivocation Region

We state our first result in the following theorem. The proof is presented in the appendix.

Theorem 1: The region $\mathcal{R}$ of achievable rates-equivocation $(K + 2)$-tuples for the source network shown in Figure 1 is the union of all $(k + 2)$-tuples $\{(R_k)_{k=1}^K, R_0, \Delta\}$ that satisfy

$$R_0 \geq I(X_1, X_2, \ldots, X_K; W), \quad (7)$$

$$R_k \geq H(X_k|W), \quad k = 1, 2, \ldots, K, \quad (8)$$

$$\Delta \leq \sum_{k=1}^K H(X_k|W) \quad (9)$$

where the union is over all auxiliary random variables $W$ arbitrarily correlated with $(X_1, X_2, \ldots, X_K)$, and where $X \equiv (X_1, X_2, \ldots, X_K)$.

Remark 2: The rate region $\mathcal{R}_{G-W}$ of the Gray-Wyner network without additional equivocation constraints is the region of $(K + 1)$ rate tuples that satisfy (7) and (8).

B. Common Information of $K$ Correlated Variables

We now present two definitions for the common information of $K$ correlated random variables.

Definition 1: The common information of $K$ correlated random variables, $C_1$, is the maximal value of $R_0$, such that $\{(R_k)_{k=1}^K, R_0, \Delta_{\text{max}}\} \in \mathcal{R}$, where

$$\Delta_{\text{max}} \triangleq \sum_{k=1}^K H(X_k|X_k).$$

Definition 2: The common information of $K$ correlated random variables, $C_2$, is the maximal value of $R_0$, such that $\{H(X_k) - R_0\}_{k=1}^K, (R_0) \in \mathcal{R}_{G-W}$.

We next state our second result.

Theorem 2: $C_1$ and $C_2$ are related as follows:

$$C_1 = C_2 = \max_{w - x_k - x_{k+1} \ldots x_K} I(X_1 X_2 \ldots X_K; W). \quad (10)$$

Proof: From Definition 1, the achievable equivocation $E$ must satisfy

$$E \geq \Delta_{\text{max}} = \sum_{k=1}^K H(X_k|X_k)$$

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These constraints are equivalent to
\[ C \text{ subject to (11), which implies that} \]
which is equivalent to the following \( K \) constraints:
\[ I(\mathcal{X} \setminus X_k; W|X_k) = 0, \quad k = 1, \ldots, K. \] (11)

Therefore, from Definition 1, \( C_1 \) is equal to the maximal \( R_0 \)
subject to (11), which implies that
\[ C_1 = \max_{W - X_k - \mathcal{X} \setminus X_k, k = 1, \ldots, K} I(X_1, \ldots, X_K; W). \]

From Definition 2, \( C_2 \) is defined as the maximal \( R_0 \)
such that \( R_k + R_0 = H(X_k) \), for \( k = 1, \ldots, K \), and
\( \{R_k\}_{k=1}^K, R_0 \) \( \in \mathcal{R}_{G-W} \). We therefore have the following constraints for \( k = 1, \ldots, K \):
\[ H(X_k) = R_k + R_0 \geq H(X_k|W) + I(X_1, \ldots, X_K; W). \] (12)
These constraints are equivalent to
\[ I(\mathcal{X} \setminus X_k; W|X_k) = 0, \quad k = 1, \ldots, K. \]

Therefore, \( C_2 \) can be written as:
\[ C_2 = \max_{W - X_k - \mathcal{X} \setminus X_k, k = 1, \ldots, K} I(X_1, \ldots, X_K; W). \]

C. Common Information: Properties

We will now develop some properties of common information of \( K \) correlated random variables defined in Theorem 2.

**Proposition 1:** The common information of \( K \) random variables, \( C(X_1, X_2, \ldots, X_K) \), is monotonically decreasing in \( K \).

**Proof:** Consider an arbitrary \( W \) satisfying the Markov chain relationship
\[ W - X_k - \mathcal{X} \setminus X_k, \quad k = 1, \ldots, K. \] (14)

First consider the following sequence of inequalities:
\[ I(X_1, \ldots, X_{K-1}, X_K; W) = I(X_1, \ldots, X_{K-1}; W) + I(X_K; W|X_1, \ldots, X_{K-1}) \geq I(X_1, \ldots, X_{K-1}; W) + I(X_2, \ldots, X_K; W|X_1) \]
\[ = I(X_1, \ldots, X_{K-1}; W) \] (17)

where (17) follows from the Markov chain relationship \( W - X_1 - (X_2, \ldots, X_K) \). Now consider the following sequence of inequalities:
\[ C(X_1, \ldots, X_K) = \max_{W - X_k - \mathcal{X} \setminus X_k, k = 1, \ldots, K} I(X_1, \ldots, X_K; W) \geq I(X_1, \ldots, X_K; W|X_1) \geq \max_{W - X_k - \mathcal{X} \setminus X_k, k = 1, \ldots, K} I(X_1, \ldots, X_K; W) \]
\[ \leq I(X_1, \ldots, X_K; W|X_1) \geq \max_{W - X_k - \mathcal{X} \setminus X_k(X_k, X_K), k = 1, \ldots, (K-1)} I(X_1, \ldots, X_{K-1}; W) \]
\[ = C(X_1, \ldots, X_{K-1}) \] (18)

where (19) follows from (17) and (20) follows from the fact that the Markov chain relationship \( W - X_k - \mathcal{X} \setminus X_k \) implies the Markov chain relationship \( W - X_k - \mathcal{X} \setminus (X_k, X_K) \). Since the random variable \( X_K \) could be chosen arbitrarily from the set \( (X_1, \ldots, X_K) \), (21) shows that the common information is monotonically decreasing in \( K \).

**Proposition 2:** \( C(X_1, X_2, \ldots, X_K) \) is upper bounded as
\[ C(X_1, X_2, \ldots, X_K) \leq \min_{i \neq j} I(X_i; X_j). \] (22)

**Proof:** We consider an arbitrary \( W \) satisfying (14), and upper bound the following mutual information:
\[ I(X_1, \ldots, X_K; W) = I(X_1; W) + I(\mathcal{X} \setminus X_1; W|X_1) \geq I(X_1; W) \]
\[ \leq I(X_1; W) + I(X_2; W|X_1) \geq I(X_1; X_2) \]
\[ = I(X_1; X_2) \]
\[ = I(X_i; X_j) \]
\[ = I(X_i; X_j) \]

where (24) follows from the Markov chain condition \( W - X_i - \mathcal{X} \setminus X_i \), and (27) follows from the Markov chain condition \( W - X_j - X_i \). The choice of \((i, j)\) was arbitrary, and therefore, the common information is upper bounded by the minimum of pairwise mutual information among all pairs, i.e.,
\[ C(X_1, \ldots, X_K) \leq \min_{i \neq j} I(X_i; X_j). \]

IV. Comparison and Examples

In [1] Wyner defines the common information of two correlated random variables \((X_1, X_2)\) as
\[ B(X_1, X_2) = \inf_{X_1 \rightarrow W \rightarrow X_2} I(X_1, X_2; W). \]

One interpretation of this common information can be obtained from the Gray-Wyner source network. The common information \( B(X_1, X_2) \) of two random variables is given as the smallest value of \( R_0 \) such that \((R_1, R_2, R_0) \in \mathcal{R}_{G-W} \) and \( R_0 + R_1 + R_2 \leq H(X_1, X_2) \). Recently, this notion of common information was generalized to \( K \) correlated random variables in [3]. The common information, \( B(X_1, \ldots, X_K) \), of \( K \) correlated random variables, as defined in [3], is given by smallest value of \( R_0 \) such that \((\{R_k\}_{k=1}^K, R_0) \in \mathcal{R}_{G-W} \) and \( R_0 + \sum_{i=1}^K R_k \leq H(X_1, \ldots, X_K) \). The common information \( B(X_1, \ldots, X_K) \) is given as
\[ B(X_1, \ldots, X_K) = \inf I(X_1, \ldots, X_K; W) \]
where the infimum is over all distributions $p(w, x_1, \ldots, x_K)$ that satisfy
\[
\sum_{w \in \mathcal{W}} p(w, x_1, \ldots, x_K) = p(x_1, \ldots, x_K)
\] (28)
\[
p(x_1, \ldots, x_K | w) = \prod_{k=1}^{K} p(x_k | w).
\] (29)

It was shown in [3] that $B(X_1, \ldots, X_K)$ is monotonically increasing in $K$. We believe that any intuitively satisfactory measure of common information should satisfy the property that the common information should decrease as the number of random variables increases. In Proposition 1, we showed that our measure of common information indeed satisfies this property.

We next prove a property of $B(X_1, \ldots, X_K)$ that helps us in comparing it with our common information $C(X_1, \ldots, X_K)$.

**Proposition 3:** $B(X_1, X_2, \ldots, X_K)$ is lower bounded as follows:
\[
\max_{i \neq j} I(X_i; X_j) \leq B(X_1, X_2, \ldots, X_K).
\] (30)

**Proof:** To prove Proposition 3, consider an arbitrary $W$ satisfying the constraints (28)-(29) and the following sequence of inequalities:
\[
I(X_1, \ldots, X_K; W) \geq I(X_i; W)
\] (31)
\[
\geq I(X_i; X_j)
\] (32)
where (32) follows from the Markov chain relationship $X_i - W - X_j$, and from the data processing inequality. In arriving at (32), the choice of $(i, j)$ was arbitrary, and therefore we can maximize over all pairs $(i, j)$ such that $i \neq j$ to get the best possible lower bound in this manner.

Using Propositions 2 and 3, we have the following:
\[
C(X_1, \ldots, X_K) \leq \min_{i \neq j} I(X_i; X_j)
\]
\[
\leq \max_{i \neq j} I(X_i; X_j) \leq B(X_1, \ldots, X_K).
\] (33)

We will now give two examples to illustrate the usefulness of our definition $C(X_1, \ldots, X_K)$ over $B(X_1, \ldots, X_K)$.

**Example 1:** Consider $K = 3$ random variables $(X_1, X_2, X_3)$ such that $X_1 \sim \text{Ber}(1/2)$, $X_2 = X_1 \oplus N$, where $N \sim \text{Ber}(\delta)$ and $X_3$ is independent of $(X_1, X_2)$. Since $X_3$ is independent of $(X_1, X_2)$, these sources have nothing in common and we should expect the ‘common information’ to be zero. Note that for these sources, $\min_{i \neq j} I(X_i; X_j) = 0$, whereas $\max_{i \neq j} I(X_i; X_j) = 1 - h(\delta)$. Therefore, from (33), we have
\[
0 \leq C(X_1, X_2, X_3) \leq 0 \leq 1 - h(\delta) \leq B(X_1, X_2, X_3),
\]
which implies that $C(X_1, X_2, X_3) = 0$, whereas $B(X_1, X_2, X_3) > 0$ for any $\delta \in (0, 1/2)$.

**Example 2:** Consider $K = 3$ random variables $(X_1, X_2, X_3)$ such that $X_1 = (X_0, X_1p)$, $X_2 = (X_0, X_2p)$ and $X_3 = (X_0, X_3p)$, where $(X_0, X_1p, X_2p, X_3p)$ are all mutually independent. Since $X_0$ appears to be the only common part in all three sources, we should expect the ‘common information’ to be equal to the entropy of $X_0$. Note that for these sources, $\min_{i \neq j} I(X_i; X_j) = \max_{i \neq j} I(X_i; X_j) = H(X_0)$. Therefore, from (33), we have
\[
0 \leq C(X_1, X_2, X_3) \leq H(X_0) \leq B(X_1, X_2, X_3),
\]
It is straightforward to show that for these sources,
\[
C(X_1, X_2, X_3) = B(X_1, X_2, X_3) = H(X_0).
\]

**Example 3:** This example shows that our measure of common information can be used to find bounds on the mutual information between any two variables in a multivariate setting.

**Proposition 4:** For a set of sources $X_1, X_2, \ldots, X_K$ that satisfy
\[
\min_{i \neq j} I(X_i; X_j) = \max_{i \neq j} I(X_i; X_j),
\] (34)
we have
\[
C(X_1, X_2, \ldots, X_K) = \min_{i \neq j} I(X_i; X_j)
\] (35)
if $B(X_1, X_2, \ldots, X_K) = \max_{i \neq j} I(X_i; X_j)$. (36)

**Proof:** The constraint (34) implies that the mutual information $I(X_i; X_j)$ is the same for all $i, j \in \{1, \ldots, K\}$, $i \neq j$. Let us start with a $W^*$ that satisfies the infimization constraints for $B(X_1, \ldots, X_K)$ and yields
\[
B(X_1, \ldots, X_K) = \max_{i \neq j} I(X_i; X_j)
\] (37)
\[
= I(X_{i_0}; X_{j_0}),
\] (38)
for some $i_0 \neq j_0$. For this $W^*$, we have
\[
I(X_{i_0}; X_{j_0}) = \max_{i \neq j} I(X_i; X_j)
\] (39)
\[
= I(X_1, \ldots, X_K; W^*)
\] (40)
\[
= I(X_{i_0}; W^*) + I(\mathcal{X} \setminus X_{i_0}; W^* | X_{i_0})
\] (41)
\[
\geq I(X_{i_0}; X_{j_0}) + I(\mathcal{X} \setminus X_{i_0}; W^* | X_{i_0})
\] (42)
where (42) follows from the fact that $W^*$ satisfies the Markov relationship $X_{i_0} - W^* - X_{j_0}$, for all $i_0 \neq j_0$. In the derivation of (42), $i_0$ can be chosen arbitrarily due to (34). Therefore, (42) implies that this $W^*$ also satisfies
\[
I(\mathcal{X} \setminus X_{i_0}; W^* | X_{i_0}) = 0
\] (43)
for all $i = 1, \ldots, K$. This in turn implies that $W^*$ serves as a valid choice in the maximization for evaluation of $C(X_1, \ldots, X_K)$. Therefore, we obtain the following lower bound for $C(X_1, \ldots, X_K)$:
\[
C(X_1, \ldots, X_K) = \max_{W - X_k \sim X_k, k = 1, \ldots, K} I(X_1, \ldots, X_K; W)
\] (44)
\[
\geq I(X_1, \ldots, X_K; W^*)
\] (45)
\[
= \max_{i \neq j} I(X_i; X_j)
\] (46)
Hence, from Proposition 1, it now follows that if $B(X_1, \ldots, X_K) = \max_{i \neq j} I(X_i; X_j)$, then
The encoder output for the private link to decoder \( b \) in bin \( J \) shown that decoder the total equivocation stated in Theorem 1. Let \( J_0 \) denote the encoder output for the public link and let \( J_k \) denote the encoder output for the private link to decoder \( k \), for \( k = 1, \ldots, K \). For \( E_k \), we have the following sequence of inequalities:

\[
E_k = \frac{1}{n} H(X_1^n, \ldots, X_k^n, X_{k+1}^n, \ldots, X_K^n | J_0, J_k)
\]

\[
= \frac{1}{n} H(X_k^n | J_0, J_k)
\]

\[
\geq \frac{1}{n} H(X_k^n | J_0, J_k) - \frac{1}{n} H(X_k^n | J_0, J_k)
\]

\[
\geq \frac{1}{n} H(X_k^n | J_0, J_k) - \epsilon_{k,n}
\]

\[
= \frac{1}{n} H(X_k^n | J_0, J_k) - \frac{1}{n} H(J_0, J_k) - \epsilon_{k,n}
\]

\[
\geq \frac{1}{n} H(X_k^n) - \frac{1}{n} H(J_0, J_k) - \epsilon_{k,n}
\]

\[
\geq \frac{1}{n} H(X_k^n) - \frac{1}{n} H(J_0) - \frac{1}{n} H(J_k) - \epsilon_{k,n}
\]

\[
\geq H(X_1, \ldots, X_K) - I(X_1, \ldots, X_K; W) - H(X_k | W)
\]

\[
\geq H(X_1, \ldots, X_K) - H(X_k | W) - \epsilon_{k,n}
\]

\[
= H(X_1, \ldots, X_K | W, X_k) - \epsilon_{k,n}
\]

\[
\geq H(X_k | W, X_k) - \epsilon_{k,n}
\]

Hence, this coding scheme yields an equivocation of \( \Delta = \sum_{k=1}^{K} H(X_k | W, X_k) \).

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