On the Synergistic Benefits of Alternating CSIT for the MISO Broadcast Channel

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Abstract—The degrees of freedom (DoF) of the two-user multiple-input single-output (MISO) broadcast channel (BC) are studied under the assumption that the form, $I_1: I_2 = 1: 2$, of the channel state information at the transmitter (CSIT) for each user’s channel can be either perfect ($P$), delayed ($D$) or not available ($N$), i.e., $I_1, I_2 \in \{P, N, D\}$, and therefore the overall CSIT can alternate between the 9 resulting states $I_1 I_2$. The fraction of time associated with CSIT state $I_1 I_2$ is denoted by the parameter $\lambda_{I_1 I_2}$ and it is assumed throughout that $\lambda_{I_1 I_2} = \lambda_{I_2 I_1}$, i.e., $\lambda_N = \lambda_{NP}, \lambda_{PD} = \lambda_{DP}, \lambda_{DN} = \lambda_{ND}$. Under this assumption of symmetry, the main contribution of this paper is a complete characterization of the DoF region of the two-user MISO BC with alternating CSIT. Surprisingly, the DoF region is found to depend only on the marginal probabilities $(\lambda_P, \lambda_D, \lambda_N) = \{I_2 \lambda_{PI_2}, I_1 \lambda_{DI_2}, \lambda_{DI_2}, \lambda_{II_2}\}, I_2 \in \{P, D, N\}$, which represent the fraction of time that any given user (e.g., user 1) is associated with perfect, delayed, or no CSIT, respectively. As a consequence, the DoF region with all 9 CSIT states, $D(I_1 I_2 : I_1, I_2 \in \{P, D, N\})$, is the same as the DoF region with only 3 CSIT states $D(\lambda_{PP}, \lambda_{DP}, \lambda_{NN})$, under the same marginal distribution of CSIT states, i.e., $(\lambda_{PP}, \lambda_{DP}, \lambda_{NN}) = (\lambda_P, \lambda_D, \lambda_N)$. The sum-DoF value can be expressed as $\text{DoF} = \min \left( \frac{3}{2} \lambda_P + 1 + \lambda_P + \lambda_D \right)$, from which one can uniquely identify the minimum required marginal CSIT fractions to achieve any target DoF value as $(\lambda_P, \lambda_D)_{\text{min}} = \left( \frac{3}{2} \lambda_P - 2, 1 - \frac{3}{2} \lambda_P \right)$ when $\text{DoF} \in \left[ \frac{3}{2}, 2 \right]$ and $(\lambda_P, \lambda_D)_{\text{min}} = (0, 0)$ when $\text{DoF} \in [0, \frac{3}{2})$. The results highlight the synergistic benefits of alternating CSIT and the tradeoffs between various forms of CSIT for any given DoF value.

Partial results are also presented for the multi-user MISO BC with $M$ transmit antennas and $K$ single antenna users. For this problem, the minimum amount of perfect CSIT required per user to achieve the maximum degrees of freedom of $\min(M, K)$ is characterized. By the minimum amount of CSIT per user, we refer to the minimum fraction of time that the transmitter has access to perfect and instantaneous CSIT from a user. Through a novel converse proof and an achievable scheme, it is shown that the minimum fraction of time perfect CSIT is required per user in order to achieve the DoF of $\min(M, K)$ is given by $\min(M, K)/K$.

Index terms—Alternating channel state information at the transmitter (CSIT), degrees of freedom, feedback, multiple-input single-output (MISO) broadcast channel.

I. INTRODUCTION

The availability of channel state information at transmitters (CSIT) is a key ingredient for interference management techniques [1]. It affects not only the capacity but also the degrees of freedom (DoF) of wireless networks. Perhaps the simplest setting that exemplifies the critical role of CSIT is the two-user vector broadcast channel, also known as the multiple input single output broadcast channel (MISO BC), in which a transmitter equipped with two antennas sends independent messages to two receivers, each equipped with a single antenna. Degrees of freedom characterizations for the MISO BC are available under a variety of CSIT models, including full (perfect and instantaneous) CSIT [2], no CSIT [3]–[6], delayed CSIT [7], [8], compound CSIT [9]–[11], quantized CSIT [12]–[14], mixed (perfect delayed and partial instantaneous) CSIT [15]–[18], asymmetric CSIT (perfect CSIT for one user, delayed CSIT for the other) [19], [20] and with knowledge of only the channel coherence patterns available to the transmitter [19], [21]. The understanding of the role of CSIT for the MISO BC is still far from complete, even from a DoF perspective, as exemplified by the Lapidoth-Shamai-Wigger conjecture [22], which is but one of the many open problems along this research avenue.

In this work we focus on an aspect of CSIT that has so far received little direct attention – that it can vary over time. Consider the MISO BC for the case in which perfect CSIT is available for one user and no CSIT is available for the other user. Incidentally, the DoF are unknown for this problem. Now, staying within the assumption of full CSIT for one user and none for the other, suppose we allow the CSIT to vary, in the sense that half the time we have full CSIT for user 1 and none for user 2, and for the remaining half of the time we have full CSIT for user 2 and none for user 1. This is one example of what we call the alternating CSIT setting. In general terms, the defining feature of the alternating CSIT problem is a joint consideration of multiple CSIT states.

We motivate the alternating CSIT setting by addressing three natural questions: 1) is it practical, 2) is it a trivial extension, and 3) is it desirable/beneficial, relative to the more commonly studied non-alternating/fixed CSIT settings?
To answer the first question, we note that alternating CSIT may be already practically unavoidable due to the time varying nature of wireless networks. However, more interestingly, the form of CSIT may also be deliberately varied as a design choice, often with little or no additional overhead. For example, acquiring perfect CSIT for one user and none for the other for half the time and then switching the role of users for the remaining half of the time, carries little or no additional overhead relative to the non-alternating case in which perfect CSIT is acquired for the same user for the entire time while no CSIT is obtained for the other user. Thus, alternating CSIT is as practical as the non-alternating CSIT setting.

The second question relates to the novelty of the alternating CSIT setting with respect to the non-alternating CSIT setting. Is the former just a direct extension of the latter? As we will show in this work, this is not the case. Surprisingly, we find that the lack of a direct relationship between the alternating and non-alternating settings works in our favor. Indeed, we are able to solve the alternating CSIT DoF problem in several cases for which the non-alternating case remains open. In particular, this includes the above mentioned case of full CSIT for one user and none for the other. As mentioned previously, for this problem, the characterization of DoF remains open in the non-alternating CSIT setting. However, we are able to find the DoF for the same problem under the alternating CSIT assumption.

The third question, whether there is a benefit of alternating CSIT relative to non-alternating CSIT, is perhaps the most interesting question. Here, we will show that the constituent fixed-CSIT settings in the alternating CSIT problem are inseparable (for more on separability, see [23]–[25]), so that the DoF of the alternating CSIT setting can be strictly larger than a proportionally weighted combination of the DoF values of the constituent fixed-CSIT settings. We call this the synergistic DoF gain of alternating CSIT. As we will show in this work, the benefits of alternating CSIT over non-alternating CSIT can be quite substantial.

Related work: In terms of the constituent fixed-CSIT schemes, this work is related to most prior studies of DoF for the MISO BC. While several recent works on mixed CSIT models, such as [15], [16], [18], also jointly consider multiple forms of CSIT, it is noteworthy that these works are fundamentally distinct as in [15], [16], [18], the multiple forms of CSIT are assumed to be simultaneously present in what ultimately amounts to a fixed-CSIT setting, as opposed to the alternating CSIT setting considered in this work. More closely related to our setting, are the recent works in [26] and [27] which involve alternating perfect and delayed CSIT models. In particular, the three receiver MISO BC with two transmit antennas is studied in [27], leading to an interesting observation that the presence of a third user, even with only two transmit antennas, can strictly increase the DoF.

Besides the two-user MISO BC, we also present some partial results for the $K$-user MISO BC with $M$ antennas at the transmitter. For this multi-user extension, we characterize the minimum amount of perfect CSIT required in order to achieve the maximum possible degrees of freedom. By minimum amount of CSIT per user, we refer to the minimum fraction of time that the transmitter has access to perfect and instantaneous CSIT from a user. Through a novel converse proof and an achievable scheme, it is shown that the minimum fraction of time, perfect CSIT is required per user in order to achieve the DoF of $\min(M, K)$ is given by $\min(M, K)/K$.

Organization: Our model of MISO broadcast channel with alternating CSIT is described in Section II. In Section III, we present the DoF region of the MISO BC under alternating CSIT and highlight several aspects and interpretations of the results. In Section IV, we present constituent encoding schemes which highlight the benefits of alternating CSIT. Achievability of the DoF region with alternating CSIT is presented in Section VII-A and the converse is presented in Section VII-B. Results for the $K$-user MISO BC are presented in Section V.

II. System Model

A two user MISO BC is considered, in which a transmitter (denoted as $Tx$) equipped with two transmit antennas wishes to send independent messages $W_1$ and $W_2$, to two receivers (denoted as $Rx_1$ and $Rx_2$, respectively), and each receiver is equipped with a single antenna. The input-output relationship is given as

$$Y(t) = H(t)X(t) + N_y(t)$$  \hspace{1cm} (1)

$$Z(t) = G(t)X(t) + N_z(t),$$  \hspace{1cm} (2)

where $Y(t)$ (resp. $Z(t)$) is the channel output at $Rx_1$ (resp. $Rx_2$) at time $t$, $X(t) = [x_1(t) \ x_2(t)]^T$ is the $2 \times 1$ channel input which satisfies the power constraint $E[||X(t)||^2] \leq P$, and $N_y(t), N_z(t) \sim \mathcal{CN}(0, 1)$ are circularly symmetric complex additive white Gaussian noises at receivers 1 and 2 respectively. The $1 \times 2$ channel vectors $H(t)$ (to receiver 1) and $G(t)$ (to receiver 2) are independent and identically distributed (i.i.d.) with continuous distributions, and are also i.i.d. over time. The rate pair $(R_1, R_2)$, with $R_i = \log(|W_i|)/n$, where $n$ is the number of channel uses, is achievable if the probability of decoding error for $i = 1, 2$ can be made arbitrarily small for sufficiently large $n$. We are interested in the degrees of freedom region $D$, defined as the set of all achievable pairs $(d_1, d_2)$ with $d_i = \lim_{P \to \infty} \frac{R_i}{\log(P)}$.

While a variety of CSIT models are conceivable, here we identify the two most important characteristics of CSIT as — 1) precision, and 2) delay. Based on these two characteristics we identify three forms of CSIT to be considered in this work.

1) Perfect CSIT (P): Perfect CSIT, or $P$, denotes those instances in which CSIT is available instantaneously and with infinite precision.

2) Delayed CSIT (D): Delayed CSIT, or $D$, denotes those instances in which CSIT is available with infinite precision but only after such delay that it is independent of the current channel state.

3) No CSIT (N): No CSIT, or $N$, denotes those instances in which no CSIT is available. The users’ channels are statistically indistinguishable in this case.

The CSIT state of user 1, $I_1$, and the CSIT state of user 2, $I_2$, can each belong to any of these three cases, $I_1, I_2 \in \{P, D, N\}$.
Specifically, the fraction of time that the state $\lambda$ occurring gives us a total of 9 CSIT states $I_1I_2 \in \{PP, PD, DP, PN, NP, DD, DN, ND, NN\}$ for the two user MISO BC. Further, let us denote by $\lambda_{I_1I_2}$ the fraction of time that the state $I_1I_2$ occurs, so that

$$\sum_{I_1I_2} \lambda_{I_1I_2} = 1. \quad (3)$$

We will assume throughout this paper that $\lambda_{I_1I_2} = \lambda_{I_2I_1}$. Specifically,

$$\lambda_{PD} = \lambda_{DP}, \quad \lambda_{PN} = \lambda_{NP}, \quad \lambda_{DN} = \lambda_{ND}. \quad (4-6)$$

This assumption is justified by the inherent symmetry of the problem, e.g., it is easy to see that if DoF were to be optimized subject to a symmetric CSIT cost constraint (the cost for acquiring CSIT state $I_1I_2$ equals the cost of $I_2I_1$) then the optimal choice of CSIT states will always satisfy the property $\lambda_{I_1I_2} = \lambda_{I_2I_1}$. Furthermore, we assume that both the receivers have perfect global channel state information.

**Problem Statement:** Given the probability mass function (pmf), $\lambda_{I_1I_2}$, the problem is to characterize the degrees-of-freedom region $D(\lambda_{I_1I_2})$.

### III. MAIN RESULTS AND INSIGHTS

Starting with the 9 parameters $\lambda_{I_1I_2}$, even if we use the 4 constraints (3)-(6) to eliminate 4 parameters (say, $\lambda_{DP}, \lambda_{NP}, \lambda_{ND}, \lambda_{NN}$), we are still left with 5 free parameters ($\lambda_{PP}, \lambda_{PD}, \lambda_{DD}, \lambda_{PN}, \lambda_{DN}$), and a challenging task of characterizing the DoF region which is a function of these 5 remaining parameters, i.e., a mapping from a region in $\mathbb{R}^5$ to a region in $\mathbb{R}^2$. While such a problem can easily become intractable or at least extremely cumbersome, it turns out — rather serendipitously — to be not only completely solvable but also surprisingly easy to describe.

#### A. Main Result

We start with the main result, stated in the following theorem.

**Theorem 1.** The DoF region $D(\lambda_{I_1I_2})$, for the two user MISO BC with alternating CSIT is given by the set of non-negative pairs $(d_1, d_2)$ that satisfy

$$d_1 \leq 1 \quad (12)$$
$$d_2 \leq 1 \quad (13)$$
$$d_1 + 2d_2 \leq 2 + \lambda_P \quad (14)$$
$$2d_1 + d_2 \leq 2 + \lambda_P \quad (15)$$
$$d_1 + d_2 \leq 1 + \lambda_P + \lambda_D \quad (16)$$

where $\lambda_P$ and $\lambda_D$ defined below denote the total fraction of time that perfect and delayed CSIT, respectively, are associated with a user:

$$\lambda_P \triangleq \lambda_{PP} + \lambda_{PD} + \lambda_{PN} \quad (17)$$
$$\lambda_D \triangleq \lambda_{DD} + \lambda_{PD} + \lambda_{DN} \quad (18)$$

Note that these two marginal fractions satisfy

$$\lambda_P + \lambda_D + \lambda_N = 1 \quad (19)$$

where $\lambda_N = \lambda_{NN} + \lambda_{PN} + \lambda_{DN}$ is the total fraction of time that no CSIT is associated with a user.

**Remark 2. [Same-Marginals Property]** From Remark 1, we make a surprising observation. Given any alternating CSIT setting considered in this work, i.e., given any $\lambda_{I_1I_2}$, there exists an equivalent alternating CSIT problem, having only three states: $PP$, $DD$ and $NN$, with fractions $\lambda_P$, $\lambda_D$, and $\lambda_N$ as defined above. The two are equivalent in the sense that they have the same DoF regions. Thus, all alternating CSIT settings considered in this work can be reduced to only symmetric CSIT states with the same marginals, without any change in the DoF region. The sum DoF as a function of $(\lambda_D, \lambda_P)$, where $\lambda_N = 1 - \lambda_P - \lambda_D$ is shown in Figure 1.

This equivalence, which greatly simplifies the representation of the DoF region, remains rather mysterious because we have
not found an argument that could establish this equivalence \emph{a priori}. The equivalence is only evident after Theorem 1 is obtained, which allows us to simplify the statement of the theorem, but does not simplify the proof of the theorem. Nevertheless, the possibility of a general relationship along these lines is intriguing.

\textbf{Remark 3. [Sum-DoF]} From (12)-(16), we can write the sum DoF as follows:

\[ d_1 + d_2 = \min \left( \frac{4 + 2\lambda_P}{3}, 1 + \lambda_P + \lambda_D \right) \]

\[ = 2 - \frac{2\lambda_N}{3} + \max(\lambda_N, 2\lambda_D), \]

(20)

where we used the fact that \( \lambda_P + \lambda_D + \lambda_N = 1 \).

\textbf{Remark 4. [Cost of Delay]} It is interesting to contrast the two different forms of CSIT, delayed versus perfect. From (20) and (21) we notice that, depending on the following condition:

\[ \lambda_D \geq \frac{\lambda_N}{2}. \]

(22)

we have two very distinct observations. We note that in the region where (22) is true, delayed CSIT is interchangeable with no CSIT, because the DoF depends only on \( \lambda_P \). Here, delay makes CSIT useless. On the other hand, in the region where \( \lambda_D < \frac{\lambda_N}{2} \), delayed CSIT is as good as perfect CSIT.

\textbf{Remark 5. [Minimum Required CSIT for a DoF value]} This tradeoff between marginal \( \lambda_P \) and \( \lambda_D \) is explicitly illustrated in Fig. 2. The most efficient point, in terms of marginal CSIT required to achieve any given value of DoF, is uniquely identified to be the bottom corner of the left most edge (highlighted corner in Fig. 2) of the corresponding trapezoid. Note that any other feasible CSIT point involves either redundant CSIT or unnecessary “instantaneous” CSIT requirements when delayed CSIT would have sufficed just as well. For example, following are the minimum CSIT requirements for various sum-DoF target values:

\[ \text{DoF} = \frac{4}{3} \Rightarrow (\lambda_P, \lambda_D) = \left( 0, \frac{1}{3} \right) \]

\[ \text{DoF} = \frac{3}{2} \Rightarrow (\lambda_P, \lambda_D) = \left( \frac{1}{4}, \frac{1}{4} \right) \]

\[ \text{DoF} = \frac{8}{5} \Rightarrow (\lambda_P, \lambda_D) = \left( \frac{2}{5}, \frac{1}{5} \right) \]

\[ \text{DoF} = \frac{5}{3} \Rightarrow (\lambda_P, \lambda_D) = \left( \frac{1}{2}, \frac{1}{6} \right) \]

\[ \text{DoF} = 2 \Rightarrow (\lambda_P, \lambda_D) = (1, 0). \]

In fact, a general expression for the minimum CSIT required to achieve a sum-DoF value is easily evaluated to be

\[ (\lambda_P, \lambda_D)_{\min} = \begin{cases} \left( \frac{2}{3} \text{DoF} - 2, 1 - \frac{1}{3} \text{DoF} \right), & \text{DoF} \in \left[ \frac{3}{2}, 2 \right] \\ (0, (\text{DoF} - 1)^+) & \text{DoF} \in [0, \frac{4}{3}] \end{cases} \]

(23)

\textbf{Remark 6. [Synergistic Gains in Asymmetric Settings]} Thereom 1 characterizes the optimal DoF region for the alternating CSIT problem under a symmetric assumption, i.e., \( \lambda_{1,2} = \lambda_{2,1} \). With this result in place, a natural question arises: are the synergistic gains provided by alternating CSIT realized only under a symmetric assumption for the fraction of states? We answer this question in the negative by showing that alternating CSIT can provide synergistic gains even when \( \lambda_{1,2} \neq \lambda_{2,1} \). As an example, suppose only two states \( PD \) and \( DP \) are present. The state \( PD \) occurs for a fraction \( \lambda_{PD} = 1/5 \), and the state \( DP \) occurs for a fraction \( \lambda_{DP} = 4/5 \), so that \( \lambda_{PD} + \lambda_{DP} = 1 \), and \( \lambda_{PD} \neq \lambda_{DP} \). Clearly, for each of these states, the optimal individual DoF is 3/2, so 3/2 is readily achievable by coding over these states separately. However, one can still use these states synergistically and achieve more than 3/2 DoF as follows: use the 5/3 DoF achieving scheme (scheme S5/3 presented in Section IV) for 2/5 fraction of time by synergistically using the state \( PD \) fully along with the state \( DP \). For the remaining 3/5 fraction of time, one can use the remaining \( DP \) state (as a separate state by itself) to achieve 3/2 DoF. The resulting achievable DoF is then:

\[ \text{DoF} = 2 \times \left( \frac{5}{3} \right) + \frac{3}{5} \times \frac{3}{2} = \frac{47}{30} > \frac{3}{2}. \]

(24)

thereby showing that even in asymmetric scenarios, alternating CSIT can provide DoF gains.

\section*{B. Synergistic Benefits}

As mentioned previously, the most interesting aspects of the alternating CSIT problem are the synergistic DoF gains. Representative examples of this phenomenon are presented next.

\textbf{Example 1:} Consider the non-alternating CSIT setting, \( PD \), in which perfect CSIT is available for one user and delayed CSIT is available for the other user. It has been shown in [20] that this setting has 3/2 DoF. Now, let us make this an alternating CSIT setting. Suppose that half of the time the CSIT is of the form \( PD \) and remaining half of the time, the CSIT is of the form \( DP \). From the main result stated in Theorem 1, it is easy to see that the optimal DoF value is now increased to 5/3. This is an example of a synergistic DoF gain from alternating CSIT. Figure 3 shows
the DoF regions corresponding to the three fixed-CSIT states – $DD$, $PD$, and $DP$; and the DoF region resulting by permitting alternation between states $PD$ and $DP$ in which each state occurs for half of the total communication period. This result also highlights the inseparability of operating over such CSIT states and shows that by jointly coding across these states, thereby collaboratively using the CSIT distributed over time, significant gains in DoF can be achieved.

**Example 2:** Another interesting example for which alternating CSIT provides provable DoF gains over non-alternating CSIT is the case when states $DD$, $PN$, and $NP$ are present. Individually, the optimal DoF for $DD$ state is $4/3$ as shown in [7]. For the $PN$ and $NP$ states, the optimal DoF value is not known; however an upper bound of $3$ is readily established. In contrast, if alternation is permitted among $DD$, $PN$, and $NP$, according to $(\lambda_{DD}, \lambda_{PN}, \lambda_{NP}) = (\frac{1}{2}, \frac{2}{3}, \frac{2}{3})$, then the optimal DoF value is $8/5$, which is larger than both $4/3$ and $3/2$, thereby showing strict synergistic gains made possible by alternating CSIT.

**Example 3:** As mentioned above, the DoF value is not known individually for fixed-CSIT state $PN$. In fact, it is our conjecture (along the lines of [22]) that for fixed CSIT state $PN$, the optimal DoF value is only $1$. However, in the alternating CSIT setting, if the states $PN$ and $NP$ are present for equal fractions of the time, then $3/2$ is the optimal DoF value.

**Example 4:** Interestingly enough, the Maddah-Ali and Tse (henceforth referred as MAT) scheme [7], or rather the alternative version of it presented in [16], may also be seen as an alternating CSIT scheme that achieves $\frac{4}{3}$ DoF with $(\lambda_{DD}, \lambda_{NN}) = (\frac{1}{3}, \frac{2}{3})$. Since the DoF of the $DD$ setting by itself is $\frac{4}{3}$ and the DoF of the $NN$ setting is $1$, and $\frac{4}{3} > \frac{1}{3}(\frac{2}{3}) + \frac{1}{3}(1)$, the synergistic gains are evident here as well. Besides the synergistic benefits, this also shows reduction in feedback requirement. For the $4/3$ achieving scheme using $DD$, $NN$ states with fraction $(\lambda_{DD}, \lambda_{NN}) = (\frac{1}{3}, \frac{2}{3})$, feedback is required only in the first $1/3$rd phase of the scheme compared to the original MAT scheme, in which feedback is required from all three phases.

We conclude this section by highlighting some of key aspects of the achievability and converse proofs. The converse proofs are inspired by the techniques developed for mixed CSIT configurations in [15] but also include some novel elements. A simple setting that highlights the novel aspects of the converse proof may be the case in which $(\lambda_{PN}, \lambda_{NP}) = (1/2, 1/2)$. For the achievability proof, the main challenge lies in identifying the core constituent schemes. In particular, core constituent schemes achieving DoF values of $3/2$, $5/3$ and $8/5$ by using minimal CSIT under various CSIT states are fundamental to the achievability of the DoF region. These constituent schemes are the topic of the next section.

**IV. CONSTITUENT SCHEMES**

In proving the achievability of the respective DoF regions, we first present so called constituent encoding schemes that form the key building blocks for the achievability of the region stated in Theorem 1. Furthermore, through these constituent encoding schemes, benefits of alternating CSIT states can be easily appreciated. These schemes are summarized in Table I.

### A. Scheme achieving 1 DoF

Achieving 1 DoF requires no CSIT; and thus any state can be used for this purpose. We denote the scheme achieving 1 DoF as follows:

- $S^1$: uses the state $NN$ and achieves $(d_1, d_2) = (1, 0)$.

### B. Scheme achieving 2 DoF

The only scheme that achieves 2 DoF corresponds to the state $PP$, i.e., when the transmitter has perfect CSIT from both receivers. This is achievable via zero-forcing. We denote this scheme as follows:

- $S^2$: uses the state $PP$ and achieves $(d_1, d_2) = (1, 1)$.

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In the table below, we summarize the constituent schemes (CS) with their corresponding DoF gains, CSIT states, and fractions of $d_1$ and $d_2$.

<table>
<thead>
<tr>
<th>Sum DoF</th>
<th>CS Notation</th>
<th>CSIT States</th>
<th>Fractions $(d_1, d_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/3</td>
<td>$S^{2/3}_1$</td>
<td>PD, DP</td>
<td>$(\frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$S^{5/3}_2$</td>
<td>DP, DD</td>
<td>$(\frac{2}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$S^{2/3}_3$</td>
<td>PD, PN, NP</td>
<td>$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$S^{5/3}_4$</td>
<td>DP, PN, NP</td>
<td>$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td>8/5</td>
<td>$S^{8/5}_5$</td>
<td>PD, NN</td>
<td>$(\frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$S^{2/3}_6$</td>
<td>DP, NN</td>
<td>$(\frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$S^{5/3}_7$</td>
<td>PN, NP</td>
<td>$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$</td>
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<tr>
<td></td>
<td>$S^{2/3}_8$</td>
<td>PN, NP</td>
<td>$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$S^{5/3}_9$</td>
<td>ND, PN</td>
<td>$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$S^{8/5}_{10}$</td>
<td>DN, NP</td>
<td>$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td>4/3</td>
<td>$S^{4/3}_{11}$</td>
<td>DD</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$S^{2/3}_{12}$</td>
<td>DD, NN</td>
<td>$(\frac{1}{2}, \frac{1}{2})$</td>
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<tr>
<td></td>
<td>$S^{5/3}_{13}$</td>
<td>DN, ND</td>
<td>$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$</td>
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<tr>
<td></td>
<td>$S^{2/3}_{14}$</td>
<td>DN, ND, NN</td>
<td>$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td>1</td>
<td>$S^1$</td>
<td>NN</td>
<td>1</td>
</tr>
</tbody>
</table>
C. Schemes achieving 4/3 DoF

The following schemes achieve 4/3 DoF:

- $S_{1/3}^4$: using DD and achieving $(d_1, d_2) = (\frac{2}{3}, \frac{2}{3})$.

This is the scheme presented in [7] and achieves sum DoF of 4/3 as follows: at $t = 1$, the transmitter sends two symbols $(u_1, u_2)$ intended for receiver 1; this step delivers a useful information symbol at receiver 1 and creates side-information at receiver 2. By a useful information symbol for receiver 1, we refer to a random linear combination of $u_1$ and $u_2$. Similarly, at $t = 2$, the transmitter sends two symbols $(v_1, v_2)$ intended for receiver 2; delivering a useful symbol at receiver 2 while creating side-information at receiver 1. Due to delayed CSIT, the transmitter can reconstruct the side-information symbols created at $t = 1, 2$. At $t = 3$, the transmitter sends a linear combination of these side-information symbols. After $t = 3$, each receiver, upon receiving this linear combination, can remove the interference by using its past overheard information. Therefore, 4/3 DoF is achievable.

- $S_{2/3}^4$: using DD, NN for fractions $(\frac{1}{3}, \frac{2}{3})$ and achieving $(d_1, d_2) = (\frac{2}{3}, \frac{2}{3})$.

We show this scheme by a modification of the MAT scheme described next. At $t = 1$, the transmitter sends $u_1 + v_1$ on the first antenna and $u_2 + v_2$ on the second antenna. Channel outputs at $t = 1$ are as follows: receiver 1 obtains $A_1(u_1, u_2) + B_1(v_1, v_2)$, whereas receiver 2 obtains $A_2(u_1, u_2) + B_2(v_1, v_2)$. Via delayed CSIT from $t = 1$, the transmitter can reconstruct $B_1(v_1, v_2)$ and $A_2(u_1, u_2)$ perfectly. At $t = 2$, it transmits $A_2(u_1, u_2)$ to both receivers using one antenna and at $t = 3$, it transmits $B_1(v_1, v_2)$ to both receivers. This scheme also achieves a DoF of 4/3. The interesting aspect is that delayed CSIT from both receivers is required only at $t = 1$; however no CSIT is required from $t = 2, 3$. Thus, by alternation between (DD, NN) for fractions $(1/3; 2/3)$, 4/3 DoF is achievable. This modification of the original MAT scheme is also mentioned in [16] and [7].

- $S_1^4$: using DN, ND for fractions $(\frac{1}{2}, \frac{1}{2})$ and achieving $(d_1, d_2) = (\frac{1}{2}, \frac{1}{2})$.

- $S_2^4$: using DN, ND, NN for fractions $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and achieving $(d_1, d_2) = (\frac{2}{3}, \frac{2}{3})$.

We now present the combined explanation of the schemes $S_{4/3}^4$ and $S_{1/3}^4$. In the original MAT scheme mentioned for $S_{1/3}^4$, after $t = 1$, the transmitter requires CSIT only from receiver 1 and at $t = 3$, the transmitter requires no CSIT. From this observation, we note that the original assumption of global delayed CSIT can be relaxed to one in which the transmitter can choose to select the available CSIT from a set of three states: state ND–no CSIT from receiver 1 and delayed CSIT from receiver 2; state DN–delayed CSIT from receiver 1 and no CSIT from receiver 2; and state NN–no CSIT from either of the receivers. If in addition, it is required that these states have to be chosen for an equal fraction (i.e., one-third) of time, then the original MAT scheme applies verbatim and 4/3 is also the optimal DoF under this alternating CSIT model with a relaxed CSIT assumption. Therefore, the schemes $S_{1/3}^4$ and $S_{4/3}^4$ also achieve a DoF of 4/3.

D. Schemes achieving 3/2 DoF

The following schemes achieve 3/2 DoF:

- $S_{1/2}^3$: using PD, NN for fractions $(\frac{1}{2}, \frac{1}{2})$ and achieving $(d_1, d_2) = (\frac{1}{2}, \frac{1}{2})$.

To show the achievability of $(d_1, d_2) = (1, \frac{1}{2})$, we show that it is possible to reliably transmit two symbols $(u_1, u_2)$ to receiver 1 and one symbol $v$ to receiver 2 in two channel uses. The CSIT configuration is chosen as PD at $t = 1$ and NN at $t = 2$. At $t = 1$, the encoder sends

$$X(1) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Bv,$$

where the $2 \times 1$ precoding vector $B$ is chosen such that $H(1)B = 0$. The outputs at the receivers at $t = 1$ are given as

$$Y(1) = H(1) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$= H_1(1)u_1 + H_2(1)u_2$$

$$\triangleq L_1(u_1, u_2)$$

$$Z(1) = G(1) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + G(1)Bv$$

$$\triangleq L_2(u_1, u_2).$$

Here, and throughout the paper, we write the $1 \times 2$ channel vectors at time $t$ as $H(t) = [H_1(t) \ H_2(t)]$ to receiver 1 and $G(t) = [G_1(t) \ G_2(t)]$ to receiver 2.

Due to delayed CSIT, the transmitter has access to $L_2(u_1, u_2)$\textsuperscript{2}. Hence, at $t = 2$, it simply sends

$$X(2) = \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix},$$

\textsuperscript{1}The channel output at receiver 1 is given by $Y_1 = L_1(u_1, u_2) + N_1(1)$. However, the additive Gaussian noise $N_1(t)$ has a variance 1, which does not scale with $P$. Therefore, the presence of such noise does not impact the DoF analysis. Therefore, for notational simplicity in the achievable schemes we have omitted the additive noise from the channel outputs.

\textsuperscript{2}Here, $L_1(u_1, u_2)$ and $L_2(u_1, u_2)$ are defined as the linear combination of the information symbols $(u_1, u_2)$ in this example and similarly throughout the paper.
so that
\[ Y(2) = H(2) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix}, \quad Z(2) = G(2) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix}. \]

(32)

Having access to \( L_1(u_1, u_2) \), along with \( L_2(u_1, u_2) \), the symbols \((u_1, u_2)\) can be decoded at receiver 1. At receiver 2, the symbol \( v \) can be decoded from \( Z(1) = L_2(u_1, u_2) + v \) by canceling out the interference \( L_2(u_1, u_2) \) which is received at \( t = 2 \). The scheme is illustrated in Figure 4.

- \( S_{3/2}^3 \): using DP, NN for fractions \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and achieving \((d_1, d_2) = \left( \frac{1}{2}, 1 \right)\).
- \( S_{3/2}^3 \): using PN, NP for fractions \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and achieving \((d_1, d_2) = \left( 1, \frac{1}{2} \right)\).

To show the achievability of \((d_1, d_2) = \left( 1, \frac{1}{2} \right)\), we show that it is possible to reliably transmit two symbols \((u_1, u_2)\) to receiver 1 and one symbol \( v \) to receiver 2 in two channel uses. The CSIT configuration is chosen as PN at \( t = 1 \) and NP at \( t = 2 \).

At \( t = 1 \), the encoder sends
\[ X(1) = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} + B(1)v, \]

where the \( 2 \times 1 \) precoding vector \( B(1) \) is chosen such that \( H(1)B(1) = 0 \). The outputs at receivers at \( t = 1 \) are given as
\[ Y(1) = H(1) \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = H_1(1)u_1, \]

(34)

and
\[ Z(1) = G(1) \begin{bmatrix} u_1 \\ 0 \end{bmatrix} + G(1)B(1)v \]

\[ \triangleq L_1(u_1, v). \]

(36)

At this point, receiver 2 requires \( u_1 \) cleanly in order to decode \( v \). At \( t = 2 \), the CSIT configuration changes to NP, and the transmitter can send \( u_1 \) cleanly to receiver 2; but at the same time it uses the second antenna to transmit \( u_2 \) which is intended for receiver 1.

\[ X(2) = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} + B(2)u_2, \]

(38)

where the \( 2 \times 1 \) precoding vector \( B(2) \) is chosen such that \( G(2)B(2) = 0 \) so that
\[ Y(2) = H(2) \begin{bmatrix} u_1 \\ 0 \end{bmatrix} + H(2)B(2)u_2 \]

\[ \triangleq L_1(u_1, u_2), \]

(39)

\[ Z(2) = G(2) \begin{bmatrix} u_1 \\ 0 \end{bmatrix} + G(2)B(2)u_2 \]

\[ = G(2) \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = G_1(2)u_1. \]

(40)

(41)

(42)

Having access to \( H_1(1)u_1 \), along with \( L'(u_1, u_2) \), the symbols \((u_1, u_2)\) can be decoded at receiver 1. At receiver 2, the symbol \( v \) can be decoded from \( Z(1) = L(u_1, v) \) by canceling out the interference \( u_1 \) which can be decoded within noise distortion at \( t = 2 \) from \( G_1(2)u_1 \). The scheme is illustrated in Figure 5.

- \( S_{3/2}^3 \): using PN, NP for fractions \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and achieving \((d_1, d_2) = \left( 1, \frac{1}{2} \right)\).
- \( S_5^3 \): using ND, PN for fractions \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and achieving \((d_1, d_2) = \left( 1, \frac{1}{2} \right)\).

To show the achievability of \((d_1, d_2) = \left( 1, \frac{1}{2} \right)\), we show that it is possible to reliably transmit two symbols \((u_1, u_2)\) to receiver 1 and one symbol \( v \) to receiver 2 in two channel uses. The CSIT configuration is chosen as ND at \( t = 1 \) and PN at \( t = 2 \). At \( t = 1 \), the encoder sends
\[ X(1) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \]

(43)

The outputs at receivers at \( t = 1 \) are given as
\[ Y(1) = H(1) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]

\[ \triangleq L_1(u_1, u_2), \]

(44)

\[ Z(1) = G(1) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]

\[ \triangleq L_2(u_1, u_2). \]

(45)

At this point, side information \( L_2(u_1, u_2) \) is created at receiver 2, and if receiver 1 can obtain \( L_2(u_1, u_2) \) cleanly, then it can decode \((u_1, u_2)\). Due to delayed CSIT from receiver 2 after \( t = 1 \), the transmitter can obtain \( L_2(u_1, u_2) \) within noise distortion.
At \( t = 2 \), the CSIT configuration changes to PN, and the transmitter can send \( L_2(u_1, u_2) \) cleanly to receiver 2; but at the same time it uses the second antenna to transmit \( v \) which is intended for receiver 2.

\[
X(2) = \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + B(2)v, \tag{46}
\]

where the \( 2 \times 1 \) precoding vector \( B(2) \) is chosen such that \( H(2)B(2) = 0 \), so that

\[
Y(2) = H(2) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + H(2)B(2)v = H_1(2)L_2(u_1, u_2), \tag{47}
\]

\[
Z(2) = G(2) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + G(2)B(2)v = G_1(2)L_2(u_1, u_2) + \alpha v. \tag{49}
\]

Having access to \( L_1(u_1, u_2) \), along with \( H_1(2)L_2(u_1, u_2) \), the symbols \( (u_1, u_2) \) can be decoded at receiver 1. At receiver 2, the symbol \( v \) can be decoded from \( Z(2) = G_1(2)L_2(u_1, u_2) + \alpha v \) by canceling out the interference \( G_1(2)L_2(u_1, u_2) \) which can be obtained within noise distortion from \( L_2(u_1, u_2) \) received at \( t = 1 \). The scheme is illustrated in Figure 6.

- \( S_3^{2/3} \): using DN, NP for fractions \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and achieving \((d_1, d_2) = \left( \frac{1}{2}, 1 \right)\).

### E. Schemes achieving 5/3 DoF

The following schemes achieve 5/3 DoF:

- \( S_3^{5/3} \): using PD, DP for fractions \( \left( \frac{2}{3}, \frac{1}{3} \right) \) and achieving \((d_1, d_2) = \left( \frac{2}{3}, 1 \right)\).

- \( S_2^{5/3} \): using DP, PD for fractions \( \left( \frac{1}{2}, \frac{2}{3} \right) \) and achieving \((d_1, d_2) = \left( \frac{1}{2}, \frac{2}{3} \right)\).

- \( S_3^{5/3} \): using PD, PN, NP for fractions \( \left( \frac{3}{5}, \frac{1}{2}, \frac{1}{3} \right) \) and achieving \((d_1, d_2) = \left( \frac{1}{2}, \frac{2}{3} \right)\).

In this scheme, we show that it is possible to reliably transmit three symbols \( (u_1, u_2, u_3) \) to receiver 1 and two symbols \( (v_1, v_2) \) to receiver 2 in a total of three channel uses. The CSIT states are chosen as PD at \( t = 1 \), PN at \( t = 2 \), and NP at \( t = 3 \). At \( t = 1 \), the encoder sends

\[
X(1) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B(1)v_1, \tag{51}
\]

where the \( 2 \times 1 \) precoding vector \( B(1) \) is chosen to satisfy \( H(1)B(1) = 0 \). The channel outputs are given as

\[
Y(1) = H(1) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{52}
\]

\[
\triangleq L_1(u_1, u_2), \tag{53}
\]

\[
Z(1) = G(1) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + G(1)B(1)v_1 \tag{54}
\]

\[
\triangleq L_2(u_1, u_2) + \alpha_1 v_1. \tag{55}
\]

Due to delayed CSIT, transmitter has access to \( G(1) \) after \( t = 1 \). It can reconstruct the interference \( L_2(u_1, u_2) \) seen at receiver 2. Hence, at \( t = 2 \), it sends

\[
X(2) = \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + B(2)v_2. \tag{56}
\]

\[
Y(2) = H(2) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + H(2)B(2)v_2 = H_1(2)L_2(u_1, u_2) + \beta u_3, \tag{57}
\]

\[
Z(2) = G(2) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + B(2)v_2 \tag{58}
\]

\[
\triangleq G_1(2)L_2(u_1, u_2) + \alpha_2 v_2. \tag{59}
\]

The key consequence of this encoding step is that receiver 2 still faces the same interference (up to a known scaling factor) as it encountered at \( t = 1 \). However, to successfully decode \( (v_1, v_2) \), it still requires this interference cleanly, i.e., it requires \( L_2(u_1, u_2) \).

The transmitter now uses the freedom provided under the alternating CSIT model and switches from CSIT state PN at \( t = 2 \) to the state NP at \( t = 3 \). Having access to \( G(3) \), it sends

\[
X(3) = \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + B(3)u_3, \tag{60}
\]

where the \( 2 \times 1 \) precoding vector \( B(3) \) is chosen such that \( G(3)B(3) = 0 \). The outputs are given as

\[
Y(3) = H(3) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} + H(3)B(3)u_3 \tag{61}
\]

\[
\triangleq H_1(3)L_2(u_1, u_2) + \beta u_3, \tag{62}
\]

\[
Z(3) = G(3) \begin{bmatrix} L_2(u_1, u_2) \\ 0 \end{bmatrix} \tag{63}
\]

\[
\triangleq G_1(3)L_2(u_1, u_2). \tag{64}
\]

Having access to \( (Y(1), Y(2), Y(3)) \), the symbols \( (u_1, u_2, u_3) \) can be decoded. Finally, upon receiving \( Z(3) \), receiver 2 successfully decodes \( (v_1, v_2) \). The scheme is illustrated in Figure 7.

- \( S_3^{5/3} \): using DP, PN, NP for fractions \( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) and achieving \((d_1, d_2) = \left( \frac{1}{3}, 1 \right)\).
F. Scheme achieving 8/5 DoF

The following scheme achieves 8/5 DoF:

- S8/5: using DD, PN, NP for fractions \((\frac{3}{5}, \frac{2}{5}, \frac{2}{5})\) and achieving \((d_1, d_2) = (\frac{3}{5}, \frac{2}{5})\).

To this end, we show that it is possible to reliably transmit 4 symbols \((u_1, u_2, u_3, u_4)\) to receiver 1, and 4 symbols \((v_1, v_2, v_3, v_4)\) to receiver 2 in a total of five channel uses. The CSIT configurations are chosen as DD, PN, NP, and NP for \(t = 1, 2, 3, 4\) and \(t = 5\) respectively. At \(t = 1\), the transmitter sends the following:

\[
X(1) = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix},
\]

so that the channel outputs are \(^3\)

\[
Y(1) = H(1) \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} = A_1(u_1, u_2) + B_1(v_1, v_2) \equiv A_1 + B_1,
\]

and

\[
Z(1) = G(1) \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} = A_2(u_1, u_2) + B_2(v_1, v_2) \equiv A_2 + B_2.
\]

Due to delayed CSIT from both receivers (the state DD at \(t = 1\)), the transmitter can reconstrcut \(B_1\) and \(A_2\) (which are the interference components at receivers 1 and 2 respectively).

At \(t = 2\), the transmitter sends \(B_1\) cleanly to receiver 1, and uses the second antenna to send \(v_3\):

\[
X(2) = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + S(2)v_3,
\]

where \(2 \times 1\) precoding vector \(S(2)\) is such that \(H(2)S(2) = 0\). The outputs at \(t = 2\) are

\[
Y(2) = H(2) \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + H(2)S(2)v_3 \cong B_1
\]

\[
Z(2) = G(2) \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + G(2)S(2)v_3 \cong B_1 + B_3,
\]

where \(B_3\) is a scaled version of \(v_3\).

At \(t = 3\), the transmitter switches the role by alternating to the NP state and sends \(A_2\) cleanly to receiver 2 and uses the second antenna to send \(u_3\). We thus have

\[
Y(3) = A_2 + A_3, \quad Z(3) = A_2.
\]

At this point, we observe that receiver 1 requires \(A_2\) and receiver 2 requires \(B_1\). Moreover, the only interference that receiver 1 has seen so far is \(B_1\); and the only interference that receiver 2 has encountered so far is \(A_2\).

\(^3\)For notational simplicity, we have used \(A_1\) to denote \(A_2(u_1, u_2)\), which is a linear combination of information symbols \((u_1, u_2)\). Similarily, \(A_2, B_1,\) and \(B_2\) are the notations used for \(A_2(u_1, u_2), B_1(v_1, v_2),\) and \(B_2(v_1, v_2)\) respectively, and should be clear from the context.

At \(t = 4\), transmitter is in the PN state and it sends \(A_2\) cleanly to receiver 1, and uses the second antenna to send \(v_4\):

\[
X(4) = \begin{bmatrix} A_2 \\ 0 \end{bmatrix} + S(4)v_4,
\]

where the \(2 \times 1\) precoding vector \(S(4)\) is chosen such that \(H(4)S(4) = 0\). The outputs at \(t = 2\) are

\[
Y(4) = H(4) \begin{bmatrix} A_2 \\ 0 \end{bmatrix} + H(4)S(4)v_4 \cong A_2
\]

\[
Z(4) = G(4) \begin{bmatrix} A_2 \\ 0 \end{bmatrix} + G(4)S(4)v_4 \cong A_2 + B_4
\]

where \(B_4\) is a scaled version of \(v_4\).

At \(t = 3\), the transmitter switches the role by alternating to the NP state and sends \(A_2\) cleanly to receiver 2 and uses the second antenna to send \(u_4\). We thus have

\[
Y(5) = B_1 + A_4, \quad Z(5) = B_1.
\]

To summarize, the channel outputs can be written as

\[
Y = \begin{bmatrix} A_1 + B_1 \\ B_1 \\ A_2 + A_3 \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} A_2 + B_2 \\ B_1 + B_3 \\ A_2 + B_4 \end{bmatrix}
\]

and \((A_1(u_1, u_2), A_2(u_1, u_2), A_3(u_3), A_4(u_4))\) (and thus \((u_1, u_2, u_3, u_4)\)) are decodable at receiver 1; and similarly \((v_1, v_2, v_3, v_4)\) are decoded at receiver 2. The scheme is illustrated in Figure 8. Thus, in order to achieve the DoF pair \((4/5, 4/5)\), interference can occupy at most one dimension in the five-dimensional output space at each receiver. This is precisely what alternation allows the transmitter to accomplish by jointly using DD, PN and NP states.

Remark 7. We note here that if the CSIT state at time \(t\) is modeled as an i.i.d. random variable, i.e., \(CSIT(t) = I_1I_2\), with probability \(\lambda_{I_1I_2}\), the corresponding DoF regions and claims would continue to hold. For instance, consider the case in which the states DD, PN, NP are present for fractions \((\frac{1}{5}, \frac{2}{5}, \frac{2}{5})\) and scheme \(S^{8/5}\) is shown to achieve 8/5 DoF.
The scheme presented above uses the state DD at \( t = 1 \) and the states PN, NP are used thereafter at \( t = 2, \ldots, 5 \). This scheme indicates that in order to achieve \( 8/5 \) DoF, the DD state should occur before the PN and NP states. Now, consider the case in which CSIT state is modeled as an i.i.d. random variable as follows:

\[
CSIT(t) = \begin{cases} 
DD & \text{w.p. } \frac{1}{2}, \\
PN & \text{w.p. } \frac{1}{5}, \\
NP & \text{w.p. } \frac{1}{10}.
\end{cases}
\]  

(81)

To substantiate the claim that \( 8/5 \) DoF is also achievable under this model, consider a long block of size \( n \). By strong typicality, as \( n \to \infty \), \( 1/5 \) of the total states would be DD states, \( 2/5 \) would be PN states and \( 2/5 \) would be PN states. Now consider a sequence of such blocks, indexed as \( b = 1, \ldots, B \). In any given block \( b \), the transmitter would use the DD states from the previous block \((b - 1)\) along with the PN, NP states from the current block which as does it for scheme \( S^{8/5} \). By letting \( B \to \infty \), this block-Markov modification of the original constituent scheme takes care of causality issues, and guarantees that the DoF claims would continue to hold if the CSIT state evolves in an i.i.d. manner over time.

V. EXTENSIONS TO THE \( K \)-USER MISO BC

The results of the two-user problem presented in Theorem 1 call for an extension to the \( K \)-user MISO BC. Such an extension would be desirable from both a theoretical and practical standpoint since the number of users in a broadcast (downlink) scenario is invariably going to be more than 2. While a comprehensive generalization to the \( K \)-user MISO BC remains elusive, nonetheless, we present partial results for the \( K \)-user MISO BC. We begin with the description of the system model and the problem statement.

A \( K \)-user MISO BC is considered in which a transmitter (with \( M \) transmit antennas) wishes to send \( K \) independent messages \( W_1, \ldots, W_K \) to \( K \) receivers, where the message \( W_k \) is intended for the \( k \)th receiver, and each receiver is equipped with a single antenna. The channel input output relationships are given as:

\[
Y_k(t) = H_k(t)X(t) + N_k(t), \quad k = 1, \ldots, K,
\]  

(82)

where \( Y_k(t) \) is the scalar channel output of receiver \( k \) at time \( t \), \( X(t) \) is the \( M \times 1 \) channel input at time \( t \) which satisfies the power constraint \( E[|X(t)|^2] \leq P \), \( N_k(t) \sim \mathcal{CN}(0, 1) \) is a circularly symmetric complex additive white Gaussian noise at receiver \( k \) at time \( t \). The \( M \times 1 \) channel vectors \( H_k(t) \) to receiver \( k \) are independent and identically distributed (i.i.d.) with continuous distributions, and are also i.i.d. over time. We assume that the receivers have global channel state information (CSIR).

From each receiver \( k \), the transmitter can have access to either perfect, delayed or no CSIT. We denote this CSIT availability for each receiver \( k \) at time \( t \) as:

\[
\mu_k(t) = \begin{cases} 
P, & \text{perfect CSIT}, \\
D, & \text{delayed CSIT}, \\
N, & \text{no CSIT}.
\end{cases}
\]  

(83)

In this paper, we consider the case in which the fraction of time the transmitter has perfect CSIT from the \( k \)th receiver is at most \( \lambda \), for all \( k = 1, \ldots, K \), i.e., for a total communication period of \( n \) channel uses, we must have

\[
\frac{\sum_{t=1}^{n} \mathbb{I}(\mu_k(t) = P)}{n} \leq \lambda,
\]  

(84)

where

\[
\mathbb{I}(\mu_k(t) = P) = \begin{cases} 
1, & \text{if } \mu_k(t) = P, \\
0, & \text{otherwise}.
\end{cases}
\]  

(85)

The rate tuple \((R_1, \ldots, R_K)\), with \( R_k = \log(|W_k|)/n \), where \( n \) is the number of channel uses, is achievable if the probability of decoding error for \( k = 1, \ldots, K \) can be made arbitrarily small for sufficiently large \( n \). The degrees of freedom region \( D(\lambda) \), is defined as the closure of the set of all achievable tuples \((d_1, \ldots, d_K)\), with \( d_k = \lim_{P \to \infty} \frac{R_k}{P} \).

Furthermore, we denote the maximum sum DoF as:

\[
\text{DoF}^*(\lambda) = \max_{\{d_i\} \in D(\lambda)} d_1 + \ldots + d_K
\]  

(86)

It is clear that \( \text{DoF}^*(\lambda) \) is a non-decreasing function of \( \lambda \) and it is upper bounded by \( \min(M, K) \). We next present the main result for this model, where we characterize the minimum value of \( \lambda \) for which \( \text{DoF}^*(\lambda) = \min(M, K) \).

**Theorem 2.** The minimum fraction of perfect CSIT per-user to achieve the maximum DoF of \( \min(M, K) \) for the \( K \)-user MISO broadcast channel is given by

\[
\lambda^*(M, K) = \begin{cases} 
0, & \min(M, K) = 1, \\
\frac{\min(M, K)}{K}, & \min(M, K) > 1.
\end{cases}
\]  

Note that Theorem 2 is trivial for \( \min(M, K) = 1 \) since with full CSIR and no CSIT, \( 1 \) DoF is achievable, and thus \( \lambda^* = 0 \). The interesting case is when \( \min(M, K) > 1 \) and we prove Theorem 1 for this case in two parts:

- We present a simple achievable scheme which utilizes a fraction of \( \lambda = \frac{\min(M, K)}{K} \) amount of CSIT per user and achieves the maximum DoF of \( \min(M, K) \). This would show that \( \lambda^*(M, N) \leq \frac{\min(M, K)}{K} \).
- We present an outer bound to the DoF region of the \( K \)-user MISO BC; which is a function of \( \lambda \). From this outer bound, we then show that \( \lambda^*(M, N) \geq \frac{\min(M, K)}{K} \).

The main contribution is the proof of the lower bound.

Also note that for \( M = K = 2 \), the result of Theorem 2 can also be obtained via Theorem 1. In particular, for \( M = K = 2 \), from Theorem 1, the sum DoF is given by:

\[
d_1 + d_2 = \min \left( \frac{4 + 2\lambda P}{3}, 1 + \lambda P + \lambda_D \right)
\]  

(87)

\[
= 2 - \frac{2\lambda N}{3} - \frac{\max(\lambda N, 2\lambda_D)}{3}.
\]  

(88)

Setting, \( d_1 + d_2 = \min(M, K) = 2 \), we obtain the necessary and sufficient condition that \( \lambda P = 1 \), which implies that in order to achieve the maximum DoF for the two-user MISO BC, perfect CSIT is required from both users for entire duration; which corresponds to \( \lambda = \frac{\min(M, K)}{K} = 1 \) and agrees with the result of Theorem 2.
A. Achieving min(M, K) DoF with \( \lambda = \min(M, K)/K \)

We first illustrate the proof through an example. Consider the case in which \( M = 2, K = 3 \) so that \( \min(M, K) = 2 \). We want to use perfect CSIT from each user for a 2/3-fraction of the total communication period. Consider the following block scheme (of block length 3) for any \( i \geq 1 \):

\[
\begin{align*}
  t = i & : \text{Perfect CSIT from Rxs 1, 2, No CSIT from Rx 3.} \\
  t = i + 1 & : \text{Perfect CSIT from Rxs 2, 3, No CSIT from Rx 1.} \\
  t = i + 2 & : \text{Perfect CSIT from Rxs 1, 3, No CSIT from Rx 2.}
\end{align*}
\]

Clearly, at each time \( t \), sum DoF of 2 is achievable by using \( M = 2 \) transmit antennas and having perfect CSIT from two distinct receivers. Thus, this scheme achieves a DoF of 2. In any given block, the number of instances transmitter obtains perfect CSIT from a receiver is 2, and the length of the block is 3. The fraction of time for which perfect CSIT is required from the 4th receiver is 2/3. We note that the scheme proposed above that requires perfect CSIT \( \lambda = 2/3 \) fraction of time is not necessarily unique. Another scheme that achieves 2 DoF has been proposed by Lee and Heath requires the following CSIT pattern\(^4\) [27]:

\[
\begin{align*}
  t = i & : \text{Delayed CSIT from receivers 1, 2, 3.} \\
  t = i + 1 & : \text{Perfect CSIT from receivers 1, 2, 3.} \\
  t = i + 2 & : \text{Perfect CSIT from receivers 1, 2, 3.}
\end{align*}
\]

We next present the proof for \( \lambda^* (M, K) \leq \min(M, K)/K \) for arbitrary \( M \) and \( K \). We consider a scheme of block length \( K \) with the following CSIT pattern:

\[
\begin{align*}
  t = 1 & : \text{Perfect CSIT from Rxs 1, 2, 3, \ldots, }\min(M, K). \\
  t = 2 & : \text{Perfect CSIT from Rxs 2, 3, 4, \ldots, }\min(M, K) + 1. \\
  t = 3 & : \text{Perfect CSIT from Rxs 3, 4, 5, \ldots, }\min(M, K) + 2. \\
  \vdots \\
  t = K & : \text{Perfect CSIT from Rxs } K, 1, 2, \ldots, \min(M, K) - 1.
\end{align*}
\]

At each time instant, perfect CSIT is present from \( \min(M, K) \) receivers and no CSIT from the remaining \( K - \min(M, K) \) receivers. A sum DoF of \( \min(M, K) \) is achievable at each time instant and therefore a sum DoF of \( \min(M, K) \) is also achievable for this scheme. The fraction of time perfect CSIT is obtained from any specific receiver is \( \min(M, K)/K \) and therefore we have shown that \( \lambda^* (M, K) \leq \min(M, K)/K \).

Figure 9 shows a useful way to interpret this scheme.

Consider a window of \( \min(M, K) \) users. The transmitter only requests perfect CSIT from the users falling in this window; and then cyclically shifts this window \( K \) times. Each user falls in the window a total number of \( \min(M, K) \) times; so that the fraction of CSIT required per user is \( \min(M, K)/K \). Note that for \( \min(M, K) = K \), there is only one such window spanning all the users and thus \( \lambda^* = 1 \), when \( \min(M, K) = K \).

B. Converse for Theorem 2

We present an outer bound to the DoF region of the \( K \)-user MISO BC in which perfect CSIT is available from the \( k \)-th user for \( \lambda \) fraction of time. For the remaining \((1 - \lambda)\) fraction of time, the transmitter could have access to either delayed CSIT or no CSIT from the \( k \)-th user. Furthermore, we make no assumptions about how the instances during which the transmitter has access to perfect/delayed/no CSIT from a specific user relate to the instances it has access to perfect/delayed/no CSIT from the remaining users. That is, the only assumption made is that from each user, the fraction of time perfect CSIT is available is \( \lambda \); and such instances could be arbitrarily distributed across the communication period.

**Lemma 1.** An outer bound for the \( K \)-user MISO BC with perfect CSIT from each user for \( \lambda \)-fraction of time is given as:

\[
M d_1 + d_2 + \ldots + d_K \leq M + (\min(M, K) - 1) \lambda \quad (89)
\]

\[
d_1 + M d_2 + \ldots + d_K \leq M + (\min(M, K) - 1) \lambda \quad (90)
\]

\[
\vdots
\]

\[
d_1 + d_2 + \ldots + M d_K \leq M + (\min(M, K) - 1) \lambda. \quad (92)
\]

We prove Lemma 1 in Section VII-C.

Summing up all the bounds in Lemma 1, we obtain:

\[
d_1 + d_2 + \ldots + d_K \leq \frac{K [M + (\min(M, K) - 1) \lambda]}{M + K - 1} \quad (93)
\]

We are interested in the case when \( d_1 + d_2 + \ldots + d_K = \min(M, K) \), and setting \( \text{DoF} = \min(M, K) \) in the l.h.s. above, we get

\[
\min(M, K) \leq \frac{K [M + (\min(M, K) - 1) \lambda]}{M + K - 1}. \quad (94)
\]

\(^4\)We note here that [27] assumes a block fading model in which the channel to the receivers remains constant for a block of \( T_c \) time slots; and the channel is known to the transmitter after \( T_f \leq T_c \) time slots. The key idea is to use the channel instances across different blocks (channels across blocks are i.i.d.) and achieve the min-cut value of 2 DoF for \( T_c = 3 \) and \( T_f = 1 \), which is equivalent to \( \lambda = 2/3 \) for our system model.
Thus, from Cases I and II, we have the following lower bound:

**Case I:** $\min(M, K) = K$

In this case, (94) simplifies to

$$K \leq \frac{K(M + (K - 1)\lambda)}{M + K - 1},$$

which leads to

$$M + (K - 1) \leq M + (K - 1)\lambda$$

which gives the bound

$$\lambda \geq 1.$$  \hspace{1cm} (97)

**Case II:** $\min(M, K) = M$

In this case, (94) simplifies to

$$M \leq \frac{K(M + (M - 1)\lambda)}{M + K - 1},$$

which leads to

$$KM + M(M - 1) \leq KM + K(M - 1)\lambda,$$

which gives the bound

$$\lambda \geq \frac{M}{K}.$$ \hspace{1cm} (100)

Thus, from Cases I and II, we have the following lower bound:

$$\lambda^*(M, K) \geq \frac{\min(M, K)}{K}. \hspace{1cm} (101)$$

This completes the proof of Theorem 2.

**VI. CONCLUSIONS**

A new model of alternating CSIT has been introduced in the context of fading broadcast channels. The DoF region has been characterized for the general alternating CSIT problem. The results highlight the benefits of configurable channel state information; and also reveal the inseparability of these channel states. In practice, the channel availability at the transmitter can vary dynamically over time and, as our results illustrate in several cases, a complete understanding of the dynamic settings can be easier than the fixed CSIT settings. For instance, the individual DoF is not known for the PN (respectively DN) setting. On the contrary, we have obtained the optimal DoF if the states PN and NP (respectively DN and ND) are both present for an equal fraction of the time. The DoF region and claims presented for the alternating CSIT problem are also applicable to the case in which the CSIT at a given time is modeled as an i.i.d. random variable, where the CSIT state at a given time is $I_1I_2$ with probability $\lambda_{I_1I_2}$. The focus of this paper has been on investigating these dynamic channel conditions and showing their benefits for the MISO broadcast channel. We believe that such scenarios are worth investigating for more complicated interference networks, such as the multi-receiver MIMO broadcast, interference and X networks.

We now consider two cases:

- **Case I:** $\min(M, K) = K$

  In this case, (94) simplifies to

  $$K \leq \frac{K(M + (K - 1)\lambda)}{M + K - 1},$$

  which leads to

  $$M + (K - 1) \leq M + (K - 1)\lambda$$

  which gives the bound

  $$\lambda \geq 1.$$  \hspace{1cm} (97)

- **Case II:** $\min(M, K) = M$

  In this case, (94) simplifies to

  $$M \leq \frac{K(M + (M - 1)\lambda)}{M + K - 1},$$

  which leads to

  $$KM + M(M - 1) \leq KM + K(M - 1)\lambda,$$

  which gives the bound

  $$\lambda \geq \frac{M}{K}.$$ \hspace{1cm} (100)

Thus, from Cases I and II, we have the following lower bound:

$$\lambda^*(M, K) \geq \frac{\min(M, K)}{K}. \hspace{1cm} (101)$$

This completes the proof of Theorem 2.

**VII. APPENDIX**

A. Achievability of $\mathcal{D}(\lambda)$

We need to show the achievability of the DoF region

$$d_1 \leq 1$$ \hspace{1cm} (102)

$$d_2 \leq 1$$ \hspace{1cm} (103)

$$d_1 + 2d_2 \leq 2 + \lambda_P$$ \hspace{1cm} (104)

$$2d_1 + d_2 \leq 2 + \lambda_P$$ \hspace{1cm} (105)

$$d_1 + d_2 \leq 1 + \lambda_P + \lambda_D.$$ \hspace{1cm} (106)

We first note that the DoF region takes two different shapes, depending on whether the $(d_1 + d_2)$ upper bound in (106) is active or not. We thus have two cases:

- **Case A:** $(d_1 + d_2)$ bound is not active. This corresponds to the following condition:

  $$\frac{2(2 + \lambda_P)}{3} \leq 1 + \lambda_P + \lambda_D,$$

  which by using $\lambda_P + \lambda_D + \lambda_N = 1$, simplifies to

  $$\lambda_N \leq 2\lambda_D.$$ \hspace{1cm} (107)

- **Case B:** $(d_1 + d_2)$ bound is active. This corresponds to the following condition:

  $$\lambda_N > 2\lambda_D.$$ \hspace{1cm} (108)

The DoF regions corresponding to the Cases A and B are shown in Figures 10 and 11 respectively. In both cases A and B, the corner points $P_1$ and $P_2$ remain fixed. We first show the achievability for $P_1$. To this end, we show the achievability of the following pair:

$$P_1 : (d_1, d_2) = (1, \lambda_P).$$ \hspace{1cm} (109)

This point can be achieved by the scheme in Table II. The achievability for the corner point $P_2$ follows due to symmetry with respect to $P_1$. In the next sub-sections, we present the achievable schemes for other corner points of Cases A and B.
1) Achievability for Case A: In this section, we show the achievability of $\mathcal{D}(\lambda)$ when $\lambda_N \leq 2\lambda_D$, which is equivalent to

$$\lambda_{NN} + \lambda_{PN} \leq 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}. \quad (110)$$

To this end, we sub-classify Case A into three mutually exclusive sub-cases as follows:

- **Case A1:**
  \[
  \lambda_{NN} \leq 2\lambda_{DD} \quad \lambda_{PN} \leq 2\lambda_{PD} + \lambda_{DN}. \quad (111) \quad (112)
  \]
  Note that (111)-(112) imply that (110) is satisfied.

- **Case A2:**
  \[
  \lambda_{NN} \leq 2\lambda_{DD} \quad \lambda_{PN} > 2\lambda_{PD} + \lambda_{DN} \quad \lambda_{NN} + \lambda_{PN} \leq 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}. \quad (113) \quad (114) \quad (115)
  \]

- **Case A3:**
  \[
  \lambda_{NN} > 2\lambda_{DD} \quad \lambda_{PN} \leq 2\lambda_{PD} + \lambda_{DN} \quad \lambda_{NN} + \lambda_{PN} \leq 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}. \quad (116) \quad (117) \quad (118)
  \]

**Remark 8.** Before proceeding, we give the intuition for the classification of Case A into the aforementioned three sub-cases. By denoting

$$L_1 \triangleq \lambda_{NN}, \quad L_2 \triangleq \lambda_{PN} \quad (119)$$

$$R_1 \triangleq 2\lambda_{DD}, \quad R_2 \triangleq 2\lambda_{PD} + \lambda_{DN}, \quad (120)$$

the condition (110) can also be interpreted as follows:

$$L_1 + L_2 \leq R_1 + R_2. \quad (121)$$

This inequality can be separately broken into pair-wise comparisons between the terms $(L_1, R_1)$ and $(L_2, R_2)$. For instance\(^5\):

- **Case A1** corresponds to $L_1 \leq R_1, L_2 \leq R_2$;
- **Case A2** corresponds to $L_1 \leq R_1, L_2 > R_2$, and $L_1 + L_2 \leq R_1 + R_2$; 
- **Case A3** corresponds to $L_1 > R_1, L_2 \leq R_2$, and $L_1 + L_2 \leq R_1 + R_2$.

**a) Case A1:** This sub-case corresponds to the following conditions:

$$\lambda_{NN} \leq 2\lambda_{DD} \quad \lambda_{PN} \leq 2\lambda_{PD} + \lambda_{DN}. \quad (122) \quad (123)$$

We next present a series of observations which form the basis for the achievable scheme.

First note that the constituent scheme $S_2^{4/3}$ uses the states DD and NN for $\frac{1}{3}$rd and $\frac{2}{3}$rd fraction of time respectively and achieves a DoF of $4/3$. The rationale for this scheme comes from the fact that the NN state itself yields DoF of 1; whereas when compensated with the DD state, it yields a higher DoF of $4/3$. This implies that if the scheme $S_2^{4/3}$ is used for a fraction $\frac{1}{2}\lambda_{NN}$ of time, then the CSIT state NN can be fully utilized/compensated using this constituent scheme. On the other hand, the fraction of time state DD gets used is $\frac{1}{2}\lambda_{NN}$ and hence for feasibility, we must have $\frac{1}{2}\lambda_{NN} \leq \lambda_{DD}$ which is precisely the condition (122). In summary, the condition (122) suggests that the state NN can be fully alternated with state DD using the scheme $S_2^{4/3}$.

A similar interpretation can be drawn from (123) as follows: the l.h.s. of (123) has the term $\lambda_{PN}$, representative of the fraction of each of PN and NP states. On the other hand, the r.h.s. has the terms consisting $\lambda_{PD}$ and $\lambda_{DN}$, representatives

\(^5\)We note that in Case A2, the inequality $L_2 > R_2$ is redundant as it follows from $L_2 > R_2$, and $L_1 + L_2 \leq R_1 + R_2$. Similarly, in Case A3, the inequality $L_2 \leq R_2$ is redundant. We however prefer to state these inequalities since these provide intuitive explanations for the selection of particular constituent schemes in achieving the corresponding DoF region.
The achievable $d_1 = d_2$ (due to symmetry) for this scheme is as follows:

$$d_1 = d_2 = \lambda_{PP} + 5 \lambda_{PD} + 2q_1 + \frac{3q_2}{2} + \frac{2}{3} (\lambda_{NN} + \lambda_{DD} + 2 \lambda_{DN} - q_2)$$

$$= \lambda_{PP} + 2 (\lambda_{NN} + \lambda_{DD}) + 5 \lambda_{PD} + 4 \lambda_{DN} + \frac{5}{3} \left( 2q_1 + q_2 \right)$$

The scheme in Table III works for any $q_1, q_2 \geq 0$ satisfying the following three conditions:

$$q_1 \leq \lambda_{PD}$$
$$q_2 \leq 2 \lambda_{DN}$$
$$2q_1 + q_2 = \lambda_{PN}.$$  

The condition (125) comes from the fact that we use the schemes $S^5/3, S^5/3$, each for $3q_1$ fraction of time. In these two schemes, the states PD and DP are used for a fraction of $q_1$ of time; and hence for feasibility, we require $q_1 \leq \lambda_{PD}$.

The condition (126) comes from the fact that we use the schemes $S^5/3, S^5/3$ each for $q_2$ fraction of time. In these two schemes, the states ND and DN are used for a fraction of $q_2$ of time; and hence for feasibility, we require $q_2 \leq 2 \lambda_{PN}$. Finally, the condition (127) ensures that the states PD and NP are fully utilized. In particular, for full utilization of the state PD, we require

$$\lambda_{PD} = \frac{q_1 + q_2}{2}.$$  

And for utilization of the state NP, which occurs for $\lambda_{PN}$ fraction of time, we require

$$\lambda_{PN} = \frac{q_1 + q_2}{2}$$

In summary, the conditions (125) and (126) ensure that the fractions of constituent schemes are non-negative; and condition (127) ensures that all fractions sum to 1 and the marginals of the original states are as desired. Note that for these to simultaneously hold, we require (123). Furthermore, condition (122) ensures that the fraction of state DD, i.e., $\lambda_{DD} - \frac{\lambda_{NN}}{2}$ is non-negative.

The achievable $d_1 = d_2$ (due to symmetry) for this scheme is as follows:

$$d_1 = d_2 = \lambda_{PP} + \frac{5}{3} (\lambda_{PD} + 2q_1) + \frac{3q_2}{2} + \frac{2}{3} (\lambda_{NN} + \lambda_{DD} + 2 \lambda_{DN} - q_2)$$

$$= \lambda_{PP} + 2 (\lambda_{NN} + \lambda_{DD}) + 5 \lambda_{PD} + 4 \lambda_{DN}$$

$$+ \frac{5}{3} \left( 2q_1 + q_2 \right) \quad \text{and} \quad \lambda_{PN} = \frac{1}{3}.$$  

The condition (134) suggests that the state NN can be fully utilized with state DD using the scheme $S_{4 \times 3}$. The condition (135) suggests that the states (PD, DP) can be fully alternated with the states (PN, NP) using schemes using $(S_{5 \times 3}, S_{4 \times 3})$ and $(S_{5 \times 3}, S_{4 \times 3})$ respectively. Finally, the condition (136) suggests that the remaining portion of (PD, DP) states can be alternated with the DD state by using the scheme $S_{8 \times 5}$.

The condition (141) suggests that the state NN can be fully utilized with state DD using the scheme $S_{4 \times 3}$. The condition (142) suggests that the states (PD, DP) can be fully utilized with the states (PN, NP) using $(S_{5 \times 3}, S_{4 \times 3})$ and with (DN,
ND) using schemes \((S_{5}^{3/2}, S_{6}^{3/2})\). Finally, the condition (143) suggests that the remaining portion of the state NN can be alternated with the set of states (PD, DP) and (DN, ND).

In Table V, we present the scheme that achieves the DoF pair corresponding to \(P_{0}\):

\[
(d_1, d_2) = \left(\frac{2 + \lambda_D}{3}, \frac{2 + \lambda_D}{3}\right)
\]

(144)

The scheme in Table V works for any \((q_1, q_2, q_3, q_4)\), with \(q_i \geq 0\) for \(i = 1, \ldots, 4\) satisfying the following conditions:

\[
q_1 + \frac{q_3}{2} \leq \lambda_{PD}
\]

(145)

\[
q_2 + 2q_4 \leq \lambda_{DN}
\]

(146)

\[
2q_1 + \frac{q_2}{2} = \lambda_{PN}
\]

(147)

\[
q_3 + q_4 = \lambda_{NN} - 2\lambda_{DD}.
\]

(148)

The conditions (145)-(146) ensure that the fractions of constituent schemes are non-negative; and conditions (147)-(148) ensure that all fractions sum to 1 and the marginals of the original states are as desired. Note that for these to hold simultaneously, we require (141)-(143).

The achievable \(d_1 = d_2\) (due to symmetry) for this scheme are as follows:

\[
d_1 = d_2 = \lambda_{PP} + \frac{5}{3} \lambda_{PD} + 2\lambda_{DD} + \frac{4}{3} \lambda_{DN}
\]

\[
+ \frac{5}{3} \left(\frac{2q_1 + q_2}{2}\right) + \frac{2}{3} \left(q_3 + q_4\right) = (\lambda_{PN} - 2\lambda_{DD})
\]

(149)

\[
= 2 + \left(\lambda_{PP} + \lambda_{PD} + \lambda_{PN}\right)
\]

(150)

\[
= \frac{2 + \lambda_P}{3}
\]

(151)

This completes the proof of achievability of \(\mathcal{D}(\lambda)\) for Case A.

2) Achievability for Case B: In this section, we show the achievability of \(\mathcal{D}(\lambda)\) when \(\lambda_N > 2\lambda_D\), which is equivalent to

\[
\lambda_{NN} + \lambda_{PN} > 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}.
\]

(152)

Similar to Case A, we sub-classify Case B into three mutually exclusive sub-cases as follows:

- **Case B1**: \(\lambda_{NN} > 2\lambda_{DD}\)
  \(\lambda_{PN} > 2\lambda_{PD} + \lambda_{DN}\).  \(\lambda_{NN} + \lambda_{PN} > 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}\).  (153)
  \(\lambda_{NN} > 2\lambda_{DD}\)
  \(\lambda_{PN} > 2\lambda_{PD} + \lambda_{DN}\).  \(\lambda_{NN} + \lambda_{PN} > 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}\).  (154)

Note that (153)-(154) imply that (152) is satisfied.

- **Case B2**:  \(\lambda_{NN} < 2\lambda_{DD}\)  \(\lambda_{PN} > 2\lambda_{PD} + \lambda_{DN}\)  \(\lambda_{NN} + \lambda_{PN} > 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}\).  (155)
  \(\lambda_{NN} > 2\lambda_{DD}\)  \(\lambda_{PN} < 2\lambda_{PD} + \lambda_{DN}\)  \(\lambda_{NN} + \lambda_{PN} > 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}\).  (156)

Here, we focus on the achievability for the corner point \(P_1\):

\[
(d_1, d_2) = (1 - \lambda_D, \lambda_P + 2\lambda_D) = (1 - \lambda_{DD} - \lambda_{PD} - \lambda_{DN}, \lambda_{PP} + 2\lambda_{DD} + 3\lambda_{PD} + 2\lambda_{DN} + \lambda_{PN}).
\]

(161)

- **a) Case B1**: This sub-case corresponds to the following conditions:
  \(\lambda_{NN} > 2\lambda_{DD}\)  \(\lambda_{PN} > 2\lambda_{PD} + \lambda_{DN}\).  (163)

The condition (163) suggests that the state DD can be fully alternated with state NN using the scheme \(S_5^{3/2}\). The condition (164) suggests that the states (PD, DP) and (DN, ND) can be fully alternated with the states (PN, NP) using the schemes \((S_5^{3/2}, S_6^{3/2})\) and \((S_5^{3/2}, S_6^{3/2})\) respectively. In Table VI, we present the scheme that achieves the DoF pair corresponding to \(P_1^*\):

\[
(d_1, d_2) = (1 - \lambda_{DD} - \lambda_{PD} - \lambda_{DN}, \lambda_{PP} + 2\lambda_{DD} + 3\lambda_{PD} + 2\lambda_{DN} + \lambda_{PN}).
\]

(165)

The achievable \(d_1, d_2\) are as follows:

\[
d_1 = \lambda_{PP} + 5\lambda_{PD} + 3\lambda_{DN} + 2(\lambda_{PN} - 2\lambda_{PD} - \lambda_{DN}) + 2\lambda_{DD} + \lambda_{NN} - 2\lambda_{DD}
\]

(166)

\[
\lambda_{PP} + \lambda_{NN} + \lambda_{DD} + 2\lambda_{PD} + 2\lambda_{PN} + 2\lambda_{DN} - \lambda_{DD} - \lambda_{PD} - \lambda_{DN}
\]

(167)

\[
= 1 - \lambda_{DD} - \lambda_{PD} - \lambda_{DN}.
\]

(168)
and

\[ d_2 = \lambda_{PP} + 5\lambda_{PD} + 3\lambda_{DN} + \lambda_{PN} - 2\lambda_{PD} - \lambda_{DN} + 2\lambda_{DD} \]
\[ = \lambda_{PP} + 2\lambda_{DD} + 3\lambda_{PD} + 2\lambda_{DN} + \lambda_{PN}. \]  

(169)

(170)

**b) Case B2:** This sub-case corresponds to the following conditions:

\[ \lambda_{NN} \leq 2\lambda_{DD} \]
\[ \lambda_{PN} > 2\lambda_{PD} + \lambda_{DN} \]
\[ \lambda_{NN} + \lambda_{PN} > 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}. \]  

(171)

(172)

(173)

The achievable \( (d_1, d_2) \) are as follows:

\[ d_1 = \lambda_{PP} + 5\lambda_{PD} + 3\lambda_{DN} + \lambda_{PN} + 2\lambda_{PN} - 2\lambda_{DN} - 4\lambda_{PD} \]
\[ = \lambda_{PP} + \lambda_{PD} + \lambda_{DN} + \lambda_{NN} + 2\lambda_{PN} \]
\[ = 1 - \lambda_{DD} - \lambda_{PD} - \lambda_{DN}. \]  

(175)

(176)

(177)

and

\[ d_2 = \lambda_{PP} + 5\lambda_{PD} + 3\lambda_{DN} + \lambda_{PN} - 2\lambda_{PD} - \lambda_{DN} + 2\lambda_{DD} \]
\[ = \lambda_{PP} + 2\lambda_{DD} + 3\lambda_{PD} + 2\lambda_{DN} + \lambda_{PN}. \]  

(178)

(179)

**c) Case B3:** This sub-case corresponds to the following conditions:

\[ \lambda_{NN} > 2\lambda_{DD} \]
\[ \lambda_{PN} \leq 2\lambda_{PD} + \lambda_{DN} \]
\[ \lambda_{NN} + \lambda_{PN} > 2\lambda_{DD} + 2\lambda_{PD} + \lambda_{DN}. \]  

(180)

(181)

(182)

The condition (180) suggests that the state DD can be fully alternated with state NN using the scheme \( S_{5/4}^{4/3} \). The condition (181) suggests that the states (PD, DP) and (DN, ND) can be fully alternated with the states (PN, NP) using the schemes \( (S_{3/4}^{5/3}, S_{4/5}^{3/4}) \) and \( (S_{5/6}^{1/1}, S_{6/5}^{1/1}) \) respectively. Finally, condition (182) suggests that the remaining DD state can be alternated with (PD, DP) and (DN, ND) states can be alternated with the state NN.

In Table VII, we present the scheme that achieves the DoF pair corresponding to \( P_1^* \):

\[ (d_1, d_2) = \left(1 - \lambda_{DD} - \lambda_{PD} - \lambda_{DN}, \lambda_{PP} + 2\lambda_{DD} + 3\lambda_{PD} + 2\lambda_{DN} + \lambda_{PN}\right). \]  

(174)
pair corresponding to $P^*_1$:

$$(d_1, d_2) = \left(1 - \lambda_{PD} - \lambda_{DN}, \lambda_{PP} + 2 \lambda_{DD} + 3 \lambda_{PD} + 2 \lambda_{DN} + \lambda_{PN}\right).$$  \hfill (183)

The scheme in Table VIII works for any choice of $(q_1, q_2)$ that satisfy $q_i \geq 0$ for $i = 1, 2$, and the following conditions:

\begin{align*}
q_1 &\leq 3 \lambda_{PD}, \quad (184) \\
q_2 &\leq 2 \lambda_{DN}, \quad (185) \\
\frac{2q_1}{3} + \frac{q_2}{2} &= \lambda_{PN}. \quad (186)
\end{align*}

The conditions (184)-(185) ensure that the fractions of the constituent schemes are non-negative. Condition (186) ensures that the states (PN, NP) are fully alternated with (PD, DP) and (DN, ND) states and the marginals of the states are preserved. This is guaranteed by condition (181).

The achievable $(d_1, d_2)$ are as follows:

\begin{align*}
d_1 &= \lambda_{PP} + \lambda_{NN} + \lambda_{PD} + \lambda_{DN} + \lambda_{PN} + \left(\frac{2q_1}{3} + \frac{q_2}{2}\right) \\
&= \lambda_{PP} + \lambda_{NN} + \lambda_{PD} + \lambda_{DN} + 2 \lambda_{PN} \quad (187) \\
&= 1 - \lambda_{DD} - \lambda_{PD} - \lambda_{DN}, \quad (188)
\end{align*}

and

\begin{align*}
d_2 &= \lambda_{PP} + 2 \lambda_{PN} + \lambda_{PD} + \lambda_{DN} + 2 \lambda_{DD} + 2 \lambda_{PD} + \lambda_{DN} - \lambda_{PN} \quad (189) \\
&= \lambda_{PP} + 2 \lambda_{DD} + 3 \lambda_{PD} + 2 \lambda_{DN} + \lambda_{PN}. \quad (190)
\end{align*}

This completes the achievability proof of $D(\lambda)$ for Case B.

### B. Converse Proofs for Theorem 1

Before proceeding to the converse proofs, we first note that the DoF outer bounds and the region in Theorem 1 remains the same even if the number of transmit antennas is greater than 2. The reason is that the total number of receive antennas is 2 (one antenna at each receiver), i.e., the total dimension of the received signal space is at most 2. Hence, without loss of generality, we can restrict the number of transmit antennas to be 2.

1) **Proof of $2d_1 + d_2$ upper bound:** We denote the channel output at the receivers as follows:

\begin{align*}
Y^n &= \left(Y^n_{pp}, Y^n_{pd}, Y^n_{dp}, Y^n_{pn}, Y^n_{pd}, Y^n_{dn}, Y^n_{dd}, Y^n_{nn}\right), \quad (192) \\
Z^n &= \left(Z^n_{pp}, Z^n_{pd}, Z^n_{dp}, Z^n_{pn}, Z^n_{pd}, Z^n_{dn}, Z^n_{dd}, Z^n_{nn}\right), \quad (193)
\end{align*}

where the subscript $Y^n_{ab}$ (respectively $Z^n_{ab}$) denotes the portion of the channel output at receiver 1 (respectively receiver 2) corresponding to the time instants that transmitter spends in state AB.

We first enhance receiver 2 by giving it the channel output of receiver 1, i.e., receiver 2 now has $(Y^n, Z^n)$. For this enhanced physically degraded broadcast channel, it is known from [28] that feedback does not increase the capacity region. Thus, we remove the delayed CSIT assumption from the states PD, DP, DN, ND and DD without effecting the capacity region.

Now, we introduce a statistically indistinguishable receiver $\tilde{1}$, which has access to the following channel output:

\begin{align*}
\tilde{Y}^n &= \left(\tilde{Y}^n_{pp}, \tilde{Y}^n_{pd}, \tilde{Y}^n_{dp}, \tilde{Y}^n_{pn}, \tilde{Y}^n_{dn}, \tilde{Y}^n_{dd}, \tilde{Y}^n_{nn}\right), \quad (194)
\end{align*}
where the channel output to receiver $\bar{1}$ is

- exactly the same as the channel output at receiver 1 corresponding to states PP, PD, PN, DN, and
- identically distributed as the channel output to receiver 1 in the states DP, NP, ND, DD and NN. In particular, channel output at the statistically indistinguishable receiver $\bar{1}$ is given by

$$\tilde{Y}_{1,i_1}(t) = \tilde{H}(t)X(t) + \tilde{N}(t),$$  \hspace{1cm} (195)$$

where $\tilde{H}(t)$ and $H(t)$ are identically distributed and independent of each other. The additive noise $\tilde{N}(t)$ is distributed as $CN(0,1)$ and independent of all other random variables.

We next note that in this enhanced broadcast channel without feedback, the capacity region depends only on the marginals. Therefore, due to this fact and due to the specific construction of the channel output to receiver $\bar{1}$, both receivers 1 and $\bar{1}$ can decode the message $W_1$. Finally, we also give the output of receiver $\bar{1}$ along with the message $W_1$ to receiver 2.

Denote $\Omega = \left(\{H(i), G(i), \tilde{H}(i)\}_{i=1}^{n}\right)$ as the global CSIT of the original broadcast channel and the CSIT of the artificial receiver $\bar{1}$ for the entire block length $n$.

We thus have the following sequence of inequalities:

$$nR_1 = H(W_1) \leq I(W_1; Y^n|\Omega) + o(n) \leq h(Y^n|\Omega) - h(Y^n|W_1, \Omega) + o(n) \leq n \log(P) - h(Y^n|W_1, \Omega) + o(n),$$  \hspace{1cm} (196)$$

where in (197), we used the independence of $W_1$ and $\Omega$, and in (198), we have used Fano's inequality to bound $H(W_1|Y^n, \Omega) \leq o(n)^6$. Similarly, for the artificial receiver $\bar{1}$, we have

$$nR_1 \leq n \log(P) - h(\tilde{Y}_n|W_1, \Omega) + o(n).$$  \hspace{1cm} (199)$$

Adding (200) and (201), we have

$$2nR_1 \leq n \log(P) - h(Y^n|W_1, \Omega) - h(\tilde{Y}_n|W_1, \Omega) + o(n)$$

$$\leq n \log(P) - h(Y^n, \tilde{Y}_n|W_1, \Omega) + o(n).$$  \hspace{1cm} (202)$$

Now, consider the enhanced receiver 2:

$$nR_2 \leq I(W_2; Z^n, Y^n, \tilde{Y}_n|W_1, \Omega) + o(n)$$

$$= h(Z^n, Y^n, \tilde{Y}_n|W_1, \Omega) - h(Z^n, Y^n, \tilde{Y}_n|W_2, \Omega) + o(n) \leq h(Z^n, Y^n, \tilde{Y}_n|W_1, \Omega) + o(n) - n \log(P)$$

$$= h(Y^n, \tilde{Y}_n|W_1, \Omega) + h(Z^n|Y^n, \tilde{Y}_n, W_1, \Omega) + o(n) - n \log(P)$$

$$\leq h(Y^n, \tilde{Y}_n|W_1, \Omega) + (Z^n|Y^n, \tilde{Y}_n, W_1, \Omega) + o(n) - n \log(P)$$

$$\leq h(Y^n, \tilde{Y}_n|W_1, \Omega) + h(Z^n|Y^n, \tilde{Y}_n, W_1, \Omega) + o(n) - n \log(P) \leq h(Y^n, \tilde{Y}_n|W_1, \Omega) + o(n) - n \log(P).$$  \hspace{1cm} (203)$$

Therefore, we have

$$2R_1 + R_2 \leq h(Y^n, \tilde{Y}_n|W_1, \Omega) + n(\lambda_{PP} + \lambda_{PD} + \lambda_{PN}) \log(P) + o(n) \leq n \log(P) + o(n)$$

$$\leq h(Y^n, \tilde{Y}_n|W_1, \Omega) + n(\lambda_{PP} + \lambda_{PD} + \lambda_{PN}) \log(P) + o(n)$$

$$\leq h(Y^n, \tilde{Y}_n|W_1, \Omega) + n(\lambda_{PP} + \lambda_{PD} + \lambda_{PN}) \log(P) + o(n)$$

$$\leq 2R_1 + R_2 \leq h(Y^n, \tilde{Y}_n|W_1, \Omega) + n(\lambda_{PP} + \lambda_{PD} + \lambda_{PN}) \log(P) + o(n) \leq n \log(P) + o(n).$$  \hspace{1cm} (204)$$

Dividing (206) by $\log(P)$, and taking the limits $n \to \infty$ and then $P \to \infty$, we have the proof for

$$2d_1 + d_2 \leq 2 + \lambda_{PP} + \lambda_{PD} + \lambda_{PN}.$$  \hspace{1cm} (207)$$

The proof for the bound on $d_1 + 2d_2$ follows in a similar manner by reversing the roles of receivers 1 and 2.

2) Proof of $d_1 + d_2$ upper bound: We next prove the bound $d_1 + d_2 \leq 1 + 2\lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + \lambda_{PN} + \lambda_{DN}$. (208)

To this end, we denote the channel outputs corresponding to channel states PP, PD, DD, collectively as follows:

$$Y_0^n = (Y_{pp}^n, Y_{pd}^n, Y_{dp}^n, Y_{dd}^n),$$  \hspace{1cm} (209)$$

$Z_0^n = (Z_{pp}^n, Z_{pd}^n, Z_{dp}^n, Z_{dd}^n).$  \hspace{1cm} (210)$$

The subscript 0 denotes the set of states \{PP, PD, PP, DD\}. With this notation in place, we can write the channel outputs at the receivers as follows:

$$Y^n = (Y_{pp}^n, Y_{pd}^n, Y_{dp}^n, Y_{dn}^n, Y_{nn}^n)$$

$$Z^n = (Z_{pp}^n, Z_{pd}^n, Z_{dn}^n, Z_{dd}^n, Z_{nn}^n).$$  \hspace{1cm} (211)$$

We next enhance the system as follows: whenever the transmitter has delayed CSIT from the receiver, we make it perfect CSIT. In particular, in the enhanced system, in the PD, DP, DD states, the transmitter now has perfect CSIT from both receivers, i.e., all four of these states are enhanced to the PP state. Similarly, the state DN is enhanced to a PN state, and the state ND is enhanced to a NP state. Note that while we
enhance the CSIT availability, the original fractions of each of these states are kept the same as they were in the original system.

Next, for each of the receivers, we introduce another statistically indistinguishable receiver, which cannot reduce the capacity region, and therefore cannot reduce the DoF. Note that now we have 4 receivers, 2 of which, say receivers 1 and 2, wish to decode the message $W_1$ and the other two receivers 2 and 2, wish to decode the message $W_2$. Furthermore, since the capacity depends only on the marginals, without loss of generality we will assume that all four receivers have the same channels in state $N$. Hence, the channel outputs at the original receivers can be changed to:

$$Y^n = (Y^n_0, Y^n_{np}, Y^n_{dp}, Y^n_{nd}, Y^n_{nn})$$
$$Z^n = (Z^n_0, Z^n_{np}, Z^n_{dp}, Z^n_{nd}, Z^n_{nn})$$

(213)

(214)

i.e., after the enhancement, the channel outputs of 1 and 2 in the NN state are the same without effecting the capacity region. Furthermore, the channel output at the artificial receivers 1 and 2 are given as:

$$\tilde{Y}^n = (\tilde{Y}^n_0, \tilde{Y}^n_{np}, \tilde{Y}^n_{dp}, \tilde{Y}^n_{nd}, \tilde{Y}^n_{nn})$$

(215)

$$\tilde{Z}^n = (\tilde{Z}^n_0, \tilde{Z}^n_{np}, \tilde{Z}^n_{dp}, \tilde{Z}^n_{nd}, \tilde{Z}^n_{nn})$$

(216)

where the relationship between the artificial channel outputs and the channel input (NP, ND states for receiver 1, and PN, DN states for receiver 2) are given as:

$$\tilde{Y}_{np}(t) = \tilde{H}(t)X_{np}(t) + \tilde{N}_1(t)$$
$$\tilde{Y}_{nd}(t) = \tilde{H}(t)X_{nd}(t) + \tilde{N}_2(t)$$
$$\tilde{Z}_{pn}(t) = \tilde{G}(t)X_{pn}(t) + \tilde{N}_1(t)$$
$$\tilde{Z}_{dn}(t) = \tilde{G}(t)X_{dn}(t) + \tilde{N}_1(t)$$

(217)

(218)

(219)

(220)

with $\tilde{H}(t), \tilde{G}(t)$ are i.i.d. with the same distribution as that of $H(t), G(t)$, and the additive noises $\tilde{N}_1(t), \tilde{N}_2(t)$ are i.i.d., with same distribution as that of $N_1(t), N_2(t)$. These channel outputs are summarized in Table IX.

For the converse, we start with an arbitrary sequence of coding schemes (indexed by $n$) that operate over $n$ channel uses, achieve rates $R_1$ and $R_2$ for the two receivers, and guarantee that $P_n \to 0$ as $n \to \infty$.

<table>
<thead>
<tr>
<th>Receiver</th>
<th>0</th>
<th>PN</th>
<th>NP</th>
<th>DN</th>
<th>ND</th>
<th>NN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$Y_0^n$</td>
<td>$Y_{pp}^n$</td>
<td>$Y_{np}^n$</td>
<td>$Y_{dp}^n$</td>
<td>$Y_{nd}^n$</td>
<td>$Y_{nn}^n$</td>
</tr>
<tr>
<td>2</td>
<td>$Z_0^n$</td>
<td>$Z_{pp}^n$</td>
<td>$Z_{np}^n$</td>
<td>$Z_{dp}^n$</td>
<td>$Z_{nd}^n$</td>
<td>$Z_{nn}^n$</td>
</tr>
</tbody>
</table>

TABLE IX

Channel outputs at original and artificial receivers.

We now prove the outer bound:

$$nR_1 \leq I(W_1; Y^n_{np}|\Omega) + o(n)$$

$$= I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nd}, Y^n_{nn}|\Omega) + o(n)$$

$$= I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nd}, Y^n_{nn}, \Omega) + o(n)$$

$$\leq n(\lambda_{PP} + 2\lambda_{PD} + \lambda_{DD}) \log(P)$$

$$+ I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nd}, Y^n_{nn}, \Omega) + o(n)$$

$$= n(\lambda_{PP} + 2\lambda_{PD} + \lambda_{DD}) \log(P) + o(n)$$

(221)

Similarly, for receiver 1, we have

$$nR_1 \leq n(\lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + \lambda_{PN} + \lambda_{DN}) \log(P)$$

$$+ I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nd}, \Omega)$$

$$- h(Y^n_{np}, Y^n_{dn}) - h(W_1, Y^n_{np}, Y^n_{dn}, \Omega)$$

$$+ o(n) + n(o(\log(P))).$$

(222)

Combining (221) and (222), we obtain

$$2nR_1 \leq 2n(\lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + \lambda_{PN} + \lambda_{DN}) \log(P)$$

$$+ 2I(W_1; Y^n_{np}, Y^n_{dn}, \Omega)$$

$$- h(Y^n_{np}, Y^n_{np}, Y^n_{dn}, Y^n_{dn}) - h(W_1, Y^n_{np}, Y^n_{dn}, \Omega)$$

$$+ o(n) + n(o(\log(P))).$$

(223)
Now consider the following term appearing in (223):

\[
[2I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nn}|Y^n, W^n_1, Y^n_{dn}, Y^n_{nn}, \Omega) - h(Y^n_{np}, Y^n_{dn}, Y^n_{dn}, Y^n_{nn}|W_1, Y^n_{dn}, Y^n_{nn}, W^n_1, \Omega)]
\]

\[
= 2I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nn}|Y^n, W^n_1) - h(Y^n_{np}, Y^n_{np}, Y^n_{dn}, Y^n_{nn}|Y^n, W^n_1, \Omega) + h(Y^n_{np}, Y^n_{dn}, Y^n_{nn}|W_1, \Omega)
\]

\[
\leq 2I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nn}|Y^n, W^n_1) - h(Y^n_{np}, Y^n_{np}, Y^n_{dn}, Y^n_{nn}|Y^n, W^n_1, \Omega) + h(Y^n_{np}, Y^n_{dn}, Y^n_{nn}|W_1, \Omega)
\]

\[
\leq 2I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nn}|Y^n, W^n_1) + 2\text{no}(\log(P))
\]

where (224) follows from the fact that the term

\[
2\text{no}(\log(P)) + h(Y^n_{np}, Y^n_{np}, Y^n_{dn}, Y^n_{nn}|W_1, \Omega)
\]

\[
- h(Y^n_{np}, Y^n_{dn}, Y^n_{nn}|W_1, W_2, \Omega)
\]

\[
\geq \text{no}(\log(P))
\]

is non-negative and (225) follows from the following facts

- \(Z^n_{np}\) can be reconstructed within noise distortion from \((Y^n_{np}, Y^n_{np})\).
- \(Z^n_{nd}\) can be reconstructed within noise distortion from \((Y^n_{nd}, Y^n_{nd})\).

(226) follows from the fact that \(W_1, W_2\) and \(\Omega\) are all mutually independent random variables, and (227) follows from the following:

\[
\lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + 2\lambda_{PN} + 2\lambda_{DN} + \lambda_{NN} = 1. \quad (228)
\]

Substituting (227) back into (223), we obtain

\[
2nR_1
\]

\[
\leq n(1 + \lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + \lambda_{PN} + \lambda_{DN})\log(P) + I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nn}|W_2, \Omega)
\]

\[
- I(W_2; Z^n_{np}, Z^n_{nd}, Y^n_{nn}|W_1, \Omega) + o(n) + 5\text{no}(\log(P)). \quad (229)
\]

Repeating the same set of arguments for receivers 2 and 2, we obtain

\[
2nR_2
\]

\[
\leq n(1 + \lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + \lambda_{PN} + \lambda_{DN})\log(P) + I(W_2; Z^n_{np}, Z^n_{nd}, Y^n_{nn}|W_1, \Omega)
\]

\[
- I(W_1; Y^n_{np}, Y^n_{dn}, Y^n_{nn}|W_2, \Omega) + o(n) + 5\text{no}(\log(P)). \quad (230)
\]

Adding (229) and (230), we obtain

\[
2n(R_1 + R_2)
\]

\[
\leq 2n(1 + \lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + \lambda_{PN} + \lambda_{DN})\log(P) + 2o(n) + 10\text{no}(\log(P)),
\]

which upon normalizing by \(2n\log(P)\) and taking the limits \(n \to \infty\) and then \(P \to \infty\) yields the upper bound

\[
d_1 + d_2 \leq 1 + \lambda_{PP} + 2\lambda_{PD} + \lambda_{DD} + \lambda_{PN} + \lambda_{DN}. \quad (231)
\]

C. Proof of Lemma 1

Since all the bounds are symmetric, it suffices to prove one of them. Therefore, we present the proof of the following bound:

\[
Md_1 + d_2 + \ldots + d_K \leq M + (\min(M, K) - 1)\lambda \quad (232)
\]

To prove this bound, we start by considering the original MISO BC. Consider the channel output \(Y^n_i\) at receiver 1 and we further expand \(Y^n_{in}\) as follows:

\[
Y^n_i = (Y^n_{in}, Y^n_{in,n}), \quad (233)
\]

where
We next arrange the channel outputs of the remaining \((K-1)\) receivers and denote these as:
\[
Y_k^n = \left( Y_{k,(1),P}^n, Y_{k,(1),NP}^n \right), \quad k = 2, \ldots, K, \tag{234}
\]
where
\[
\begin{align*}
Y_{k,(1),P}^n: & \text{ channel outputs at receiver } k \text{ for those time instances in which perfect CSIT is present from receiver } 1, \\
Y_{k,(1),NP}^n: & \text{ channel outputs at receiver } k \text{ for those time instances in which perfect CSIT is not present from receiver } 1.
\end{align*}
\]

We next enhance the original BC by colluding the outputs of receivers 2, \ldots, \(K\). That is, we now have two receivers: receiver 1 and receiver 2 with the following channel outputs:
\[
\begin{align*}
&\text{At receiver 1: } (Y_{1,P}^n, Y_{1,NP}^n) \\
&\text{At receiver 2: } \\
&\left( Y_{2,(1),P}^n, \ldots, Y_{K,(1),P}^n, Y_{1,NP}^n, Y_{2,(1),NP}^n, \ldots, Y_{K,(1),NP}^n \right)
\end{align*}
\]

We further enhance this BC by giving the channel output of receiver 1 to receiver 2:
\[
\begin{align*}
&\text{At receiver 1: } (Y_{1,P}^n, Y_{1,NP}^n) \\
&\text{At receiver 2: } \\
&\left( Y_{2,(1),P}^n, \ldots, Y_{K,(1),P}^n, Y_{1,NP}^n, Y_{2,(1),NP}^n, \ldots, Y_{K,(1),NP}^n \right)
\end{align*}
\]

This is a physically degraded broadcast channel, for which it is known from [28] that feedback does not increase the capacity region (and hence the DoF region). Therefore, from the instances corresponding to \(Y_{1,NP}^n\), we can remove the assumption of delayed CSIT from receiver 1 (if any) without effecting the capacity (and the DoF) region.

We next introduce \((M-1)\) artificial receivers that are statistically indistinguishable from receiver 1. We denote the outputs at the artificial receiver \(j\) as follows:
\[
\begin{align*}
\hat{Y}_j^n = \left( Y_{1,P}^n, \hat{Y}_{j,NP}^n \right), \quad j = 1, \ldots, M-1, \tag{235}
\end{align*}
\]

where \(Y_{1,P}^n\) is exactly the same as the channel output at receiver 1 corresponding to instances with perfect CSIT from receiver 1; and the channel corresponding to \(\hat{Y}_{j,NP}^n\) is identically distributed as the channel output to receiver 1 as in \(Y_{1,NP}^n\).

Let \(\Omega\) denote the total channel state information of the original BC and that of the artificial receivers. We next note that in this enhanced physically degraded broadcast channel without feedback, the capacity region depends only on the marginals and thus if receiver 1 can decode the message \(W_1\); then all the artificial \(M-1\) receivers must also be able to decode the message \(W_1\).

We have the following sequence of bounds for receiver 1:
\[
\begin{align*}
nR_1 &= H(W_1) = H(W_1|\Omega) \\
&\leq I(W_1; Y_{1,NP}^n|\Omega) + \text{no}(n) \tag{236} \\
&= I(W_1; Y_{1,P}^n, Y_{1,NP}^n|\Omega) + \text{no}(n) \tag{237} \\
&= h(Y_{1,P}^n, Y_{1,NP}^n|\Omega) - h(Y_{1,P}^n, Y_{1,NP}^n|W_1, \Omega) + \text{no}(n) \tag{238} \\
&\leq n \log(P) - h(Y_{1,P}^n, Y_{1,NP}^n|W_1, \Omega) + \text{no}(n) \tag{239} \\
&= n \log(P) + \text{no}(n) \\
&- h(Y_{1,P}^n|W_1, \Omega) - h(Y_{1,NP}^n|W_1, Y_{1,P}^n, \Omega). \tag{240}
\end{align*}
\]

Similarly, for each of the artificial receiver \(j\), we also have:
\[
\begin{align*}
nR_1 &\leq n \log(P) + \text{no}(n) \\
&\leq h(Y_{1,P}^n|W_1, \Omega) - h(\hat{Y}_{j,NP}^n|W_1, Y_{1,P}^n, \Omega) \tag{241} \\
&\leq n \log(P) - h(\hat{Y}_{j,NP}^n|W_1, Y_{1,P}^n, \Omega) + \text{no}(n) - \text{no}(\log(P)). \tag{242}
\end{align*}
\]

Adding these total \(M\) bounds, we obtain
\[
\begin{align*}
nMR_1 &\leq nM \log(P) - h(Y_{1,P}^n|W_1, \Omega) \\
&- h(Y_{1,NP}^n|W_1, Y_{1,P}^n, \Omega) \\
&\quad - \sum_{j=1}^{M-1} h(\hat{Y}_{j,NP}^n|W_1, Y_{1,P}^n, \Omega) + \text{no}(M\alpha(n) - \text{no}(\log(P))) \tag{243} \\
&\leq nM \log(P) - h(Y_{1,P}^n|W_1, \Omega) \\
&- h(Y_{1,NP}^n, \hat{Y}_{1,NP}^n, \ldots, \hat{Y}_{M-1,NP}^n|W_1, Y_{1,P}^n, \Omega) + \text{no}(M\alpha(n) - \text{no}(\log(P))). \tag{244}
\end{align*}
\]

We next have the following sequence of bounds for the remaining receivers 2, \ldots, \(K\):
\[
\begin{align*}
n(R_2 + \cdots + R_K) \\
&= H(W_2, \ldots, W_K) \\
&= H(W_2, \ldots, W_K|W_1, \Omega) \tag{245} \\
&\leq I(W_2, \ldots, W_K; Y_{2,1}^n, \ldots, Y_{K,1}^n, \hat{Y}_{1,NP}^n, Y_{M-1,1}^n|W_1, \Omega) + \text{no}(n) \tag{246} \\
&\leq h(Y_{2,1}^n, \ldots, Y_{K,1}^n, \hat{Y}_{1,NP}^n, Y_{M-1,1}^n|W_1, \Omega) + \text{no}(n) \tag{247} \\
&= h(Y_{1,1}^n, \hat{Y}_{1,NP}^n, Y_{M-1,1}^n|W_1, \Omega) + \text{no}(n) \tag{248} \\
&= h(Y_{2,1}^n, \ldots, Y_{K,1}^n, \hat{Y}_{M-1,1}^n|W_1, \Omega) \tag{249} \\
&\leq h(Y_{1,1}^n, \hat{Y}_{1,NP}^n, Y_{M-1,1}^n|W_1, \Omega) + \text{no}(n) \tag{250} \\
&+ h(Y_{2,1}^n, \ldots, Y_{K,1}^n, \hat{Y}_{M-1,1}^n, W_1, \Omega) \tag{251}
\end{align*}
\]
\[ h(Y^n_{1,NP} \mid W_1, \Omega) + no(n) \]

\[ + h(Y^n_{1,1,NP} \mid Y^n_{1,n,1-NP}, \cdots, Y^n_{M-1,1-NP} \mid W_1, Y^n_{1,1,NP}, \Omega) \]

\[ + h(Y^n_{2,(1,1-NP)} \mid Y^n_{2,1-NP}, \cdots, Y^n_{2,M-K,(1,1-NP)} \mid Y^n_{1,1-NP}, \cdots, Y^n_{M-1,1-NP}, \Omega) \]

\[ \leq n\log(P) \] (252)

In (252), we used the fact that given \( Y^n_{1,NP} \) and \( \Omega \), the channel input \( X^n_{1,NP} \) can be obtained within noise distortion. Adding (245) and (252), we obtain

\[ n[M \lambda_1 + R_2 + \cdots + R_K] \leq nM \log(P) + h\left( Y^n_{2,(1,1-NP)} \mid Y^n_{1,1-NP}, \Omega \right) \]

\[ + no(n) + no(\log(P)). \] (253)

We next note that each of the \((K - 1)\) sequences \( Y^n_{2,(1,1-NP)}, \cdots, Y^n_{K,(1,1-NP)} \) are of length at most \( \lambda n \). We proceed to upper bound the second term in (253) by considering two cases:

If \( \min(M, K) = K \), then we have the following bound:

\[ h\left( Y^n_{2,(1,1-NP)} \mid Y^n_{1,1-NP}, \Omega \right) \]

\[ \leq \sum_{i=1}^{\lambda n} h(Y^n_{2,(1,1-NP)}(i), \cdots, Y^n_{K,(1,1-NP)}(i)) \] (254)

\[ \leq \sum_{i=1}^{\lambda n} \sum_{j=2}^{K} h(Y^n_{j,(1,1-NP)}(i)) \] (255)

\[ \leq n(K - 1) \lambda \log(P) \] (256)

If \( \min(M, K) = M \), then we proceed as follows:

\[ h\left( Y^n_{2,(1,1-NP)} \mid Y^n_{1,1-NP}, \Omega \right) \]

\[ \leq \sum_{i=1}^{\lambda n} h(Y^n_{2,(1,1-NP)}(i), \cdots, Y^n_{K,(1,1-NP)}(i) \mid Y^n_{1,1-NP}, \Omega) \] (257)

\[ = \sum_{i=1}^{\lambda n} h(Y^n_{2,(1,1-NP)}(i), \cdots, Y^n_{M,(1,1-NP)}(i) \mid Y^n_{1,1-NP}, \Omega) \]

\[ + \sum_{i=1}^{\lambda n} h(Y^n_{M+1,(1,1-NP)}(i), \cdots, Y^n_{K,(1,1-NP)}(i) \mid Y^n_{1,1-NP}, \Omega) \]

\[ + \cdots + Y^n_{M,(1,1-NP)}(i) \mid Y^n_{1,1-NP}, \Omega) \] (258)

\[ \leq n(M - 1) \lambda \log(P) + n\lambda o(\log(P)). \] (259)

Thus, the channel input \( X^n_{1,1-NP}(i) = [X^n_{1,1-NP}(i), \cdots, X^n_{M,1-NP}(i)]^T \) at time \( i \) can be obtained via channel inversion. Subsequently \( Y^n_{M+1,(1,1-NP)}(i), \cdots, Y^n_{K,(1,1-NP)}(i) \) can obtained within noise distortion from \( X^n_{1,1-NP}(i) \) and \( \Omega \). Therefore, from (256) and (259), we conclude that

\[ h\left( Y^n_{2,(1,1-NP)} \mid Y^n_{1,1-NP}, \Omega \right) \leq n(\min(M, K) - 1) \lambda \log(P) + n\lambda o(\log(P)), \] (261)

Hence, from (253) and (261), we have

\[ MR_1 + R_2 + \cdots + R_K \leq [M + (\min(M, K) - 1) \lambda] \log(P) + o(n) + o(\log(P)) \] (262)

Normalizing by \( \log(P) \) and taking the limits \( n \to \infty \) and then \( P \to \infty \), we obtain

\[ M d_1 + d_2 + \cdots + d_K \leq M + (\min(M, K) - 1) \lambda. \] (263)

This completes the proof of Lemma 1.

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REFERENCES


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