System Parameters Affecting LWA Calibration (Memo 52 Redux)

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1 Introduction

In Memo 52 ("Estimated Collecting Area Needed for LWA Calibration") Aaron Cohen attempted to estimate the number of stands needed in an LWA station to ensure the calibratibility of the instrument in the goal 52-station configuration. Using the best information available at that time and extrapolating from VLA 74 MHz experience, he estimated 176 stands/station considering only zenith pointing at 74 MHz. This result appears to be the primary justification for the current "consensus specification" of 256 stands/station (wanting to be conservative and 256 being the nearest power of 2 greater than 176).

This memo repeats Aaron's analysis, attempting to remove or explicitly parameterize as many assumptions as possible, and leaving frequency, zenith angle, and receiver temperature as independent variables. Aaron's original 74 MHz broadside estimate is found to be quite reasonable, but it also found that significantly more than 256 stands/station are required for lower zenith angles and at other frequencies. For example, to access a zenith angle of 74° (e.g., for Galactic Center work) at 74 MHz may require somewhere between 400 and 1500 stands, depending on calibratibility assumptions. Other "bottom line" results are summarized in Figure 3 and 4.

It must be emphasized that this work is only an incremental evolution of the Memo 52 work and is still limited by some fairly onerous assumptions. Mutual coupling is not rigorously treated and could conceivably change results by as much as 35% in either direction. Collecting area is also only superficially considered and could be the source of errors on the order of 10's of percent. There is also great uncertainty about what is truly required for calibratibility, which of course also varies with the calibration techniques employed and ionospheric conditions. Thus, this work should be revisited from time to time as improved information becomes available.

Acknowledgement: Many thanks to Aaron Cohen for guiding the author through some of the subtleties of full-field calibration and his previous work on this topic.

2 LWA Technical Characteristics

2.1 Image Sensitivity

The RMS noise level σ in an LWA image is given by

$$\sigma = \frac{2kT_{sys}}{\eta_s A_{es}\sqrt{N_S(N_S-1)N_{pol}\Delta\tau\Delta\nu}} \quad (1)$$

where:

- $k = 1.38 \times 10^{-23}$ [J/K]
- η_s is "system efficiency", proposed by Aaron in [1], which accounts for the aggregate effect of various hard-to-characterize losses "due to the correlator and the electronics". Aaron suggests a value of 0.78 based on VLA experience.
- A_{es} is the effective collecting area of a station.
- N_S is the number of stations; nominally 52 for the goal instrument and something like 16 for a fully-operational but intermediate phase of development. For earlier phases of development for which $N_S < 16$ or so "full field" calibration is not possible. In this case we're back to using a few bright sources with self-calibration as described in Memo 80 [2], and in this case the calibratibility requirements discussed in subsequent sections do not apply.
- N_{pol} is the number of orthogonal polarizations; nominally 2.
- $\Delta\tau$ is the total observation time; [1] suggests 6 s as a typical value.

- $\Delta\nu$ is the observed bandwidth. [1] used 4 MHz, however recent science considerations [5] suggest the right number is about 8 MHz.

The effective collecting area of a station is given by

$$A_{es} = \gamma N_a A_e \tag{2}$$

where:

- A_e is the collecting area of a single element, measured at broadside (i.e., towards zenith) and in isolation from any other elements.
- N_a is the number of stands in the station, where a stand is defined as the combination of 2 elements with presumably orthogonal polarizations. (Note that the fact that a stand consists of 2 polarizations is taken into account by the value N_{pol} in Equation 1.)
- γ is a coefficient which accounts for the aggregate effect of mutual coupling. It is shown in Memo 73 [3] that γ is in the range $1 \pm 35\%$ (variation with respect to θ and ϕ) for a station consisting of straight dipoles near resonance at 38 MHz. This may or may not also be the case for a station consisting of LWA candidate antennas at this or other frequencies.

The effective collecting area of an element is given by

$$A_e = A_{e0}(\lambda, \theta, \phi) L_g \xi \tag{3}$$

where:

- $A_{e0}(\lambda, \theta, \phi)$ is the effective collecting area of an isolated element over a perfectly-conducting ground screen (thus, no ground loss) assuming perfectly impedance-matched conditions. Note that this is a function of frequency, zenith angle, and also orientation relative to the E- and H-planes of the element. This depends somewhat sensitively on the design of the element, and thus we shall leave it as a free parameter for the time being. However, we shall revisit this in Section 2.2.
- L_g is ground loss, which is essentially zero ($L_g = 1$) if a ground screen is used. A typical value for untreated soil is $L_g = 0.66$ [3].
- ξ is impedance mismatch efficiency; that is, the fraction of power captured by the antenna which is successfully transmitted to the load. This is given by $\left(1 - |\Gamma|^2\right)$ where Γ is the reflection coefficient for the antenna-active balun interface.

The system temperature is given by

$$T_{sys} = \xi T_{sky} + \mu T_p \tag{4}$$

where:

- T_p is the noise temperature of the receiver, which is nominally dominated by the noise temperature of the preamp (active balun) following the antenna.
- T_{sky} is the antenna temperature, which is nominally dominated by the Galactic background. In this case, $T_{sky} \approx T_{74} \left(\lambda/4 \text{ m}\right)^{2.6}$ where T_{74} is defined to be 2000 K.
- μ accounts for the possibility that T_p depends on input match or other factors. We have traditionally assumed $\mu = 1$; it may be less when the match to the antenna is good, due to the possibility of noise egress out the antenna in this case. In any event, $\mu = 1$ is probably conservative.

Making the substitutions we have:

$$\sigma = B N_a^{-1} \left[T_{74} \left(\frac{\lambda}{4 \text{ m}} \right)^{2.6} + \frac{\mu}{\xi} T_p \right] \tag{5}$$

where

$$B \equiv \frac{2k}{\eta_s \gamma A_{e0}(\lambda, \theta, \phi) L_g \sqrt{N_S (N_S - 1) N_{pol} \Delta\tau \Delta\nu}} \tag{6}$$

Frequency	Zenith Gain (G_T)
20 MHz	8.4 dBi
38 MHz	7.9 dBi
74 MHz	6.5 dBi
88 MHz	5.9 dBi

Table 1: Zenith gain of the "mLWDA" big blade type antenna from Memo 32. Assumes perfectly conducting ground.

2.2 Collecting Area of a Single Element

In the derivation above, $A_{e0}(\lambda, \theta, \phi)$ is left as a free parameter. Recall that this parameter is the collecting area of a single element in isolation, over a perfectly conducting screen, assuming perfect impedance matching. There are two ways we can deal with this parameter.

The first is to obtain the actual values, either through measurements or simulation. In this case, it may be desirable to leave $L_g A_{e0}(\lambda, \theta, \phi)$ as the free parameter, i.e., include ground loss. Even better would be to leave $\gamma L_g A_{e0}(\lambda, \theta, \phi)$ as the free parameter, performing the measurement or simulation of the entire station array and dividing by N_a to get the desired per-element values. If either of these strategies are used, then one must of course be sure to remove the factors of γ and/or L_g from subsequent analyses.

Measurement or simulation is not always practical or desirable. For this reason we now develop a reasonably simple model. LWA Memo 32 [4] derives the characteristics of a "big blade" type LWA candidate antenna from which we may extract parameters suitable for our simple model. In Memo 32, this antenna is described as the "mLWDA," as it is essentially an LWDA antenna that has been scaled up by a factor of 1.37. This antenna is used here because it is representative of the class of the antenna currently under consideration, and also because the author has detailed documentation and a validated electromagnetic (NEC-2) model already worked out for this design. The antenna impedance for the mLWDA antenna is shown in Figure 1, and the zenith gain G_T is tabulated in Table 1. The subscript "T" is used to emphasize that this gain is computed in "transmit mode"; i.e., by applying a test voltage to the antenna terminals and calculating the resulting power transmitted into the far field. This may underestimate the gain at frequencies above first resonance due to the possibility of additional significant current modes present in the receive case which are not stimulated in the transmit case. Preliminary investigation has suggested that this difference can be on the order of 10% or so.

To complete the model we include a zenith angle dependence to account for the fact that collecting area decreases with increasing zenith angle due to the pattern of the antenna. A common strategy is to assume the relevant factor is $\cos^\alpha \theta$. Examination of the 38 MHz mLWDA patterns in Memo 32 suggests $\alpha = 1.34$ in the E-plane and $\alpha = 1.88$ in the H-plane. Of course, the expression for sensitivity derived earlier does not make this distinction; the number of polarizations per stand is accounted for elsewhere and so the collecting area of interest is really some combination of the available polarizations. For this reason, $\alpha = 1.6$ (the geometric mean of 1.34 and 1.88) is suggested. The complete model becomes:

$$A_{e0}(\lambda, \theta) = G_T(\lambda)\, \frac{\lambda^2}{4\pi}\, \cos^{1.6}\theta \tag{7}$$

This model is probably pretty good below about 65 MHz. At higher frequencies the pattern becomes complex; by 74 MHz a small deviation from the simple cosine power law is apparent, and by 88 MHz the E-plane severely distorted.

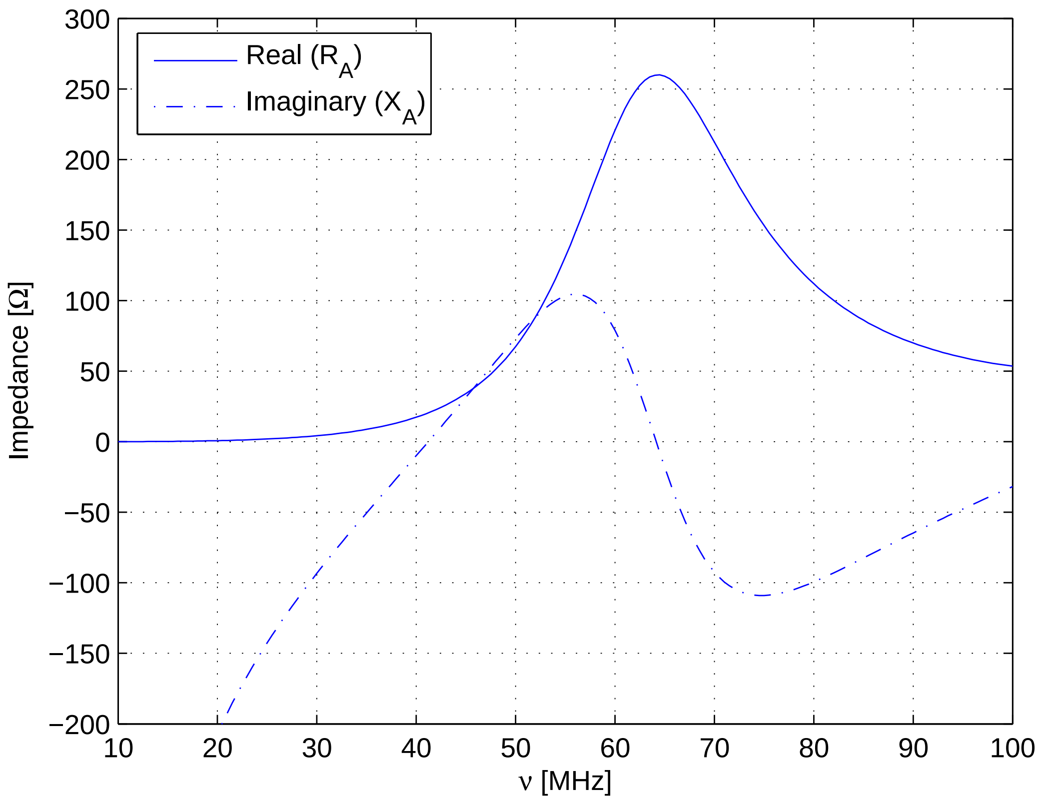

Figure 1: Impedance of the “mLWDA” big blade type antenna from Memo 32 [4].

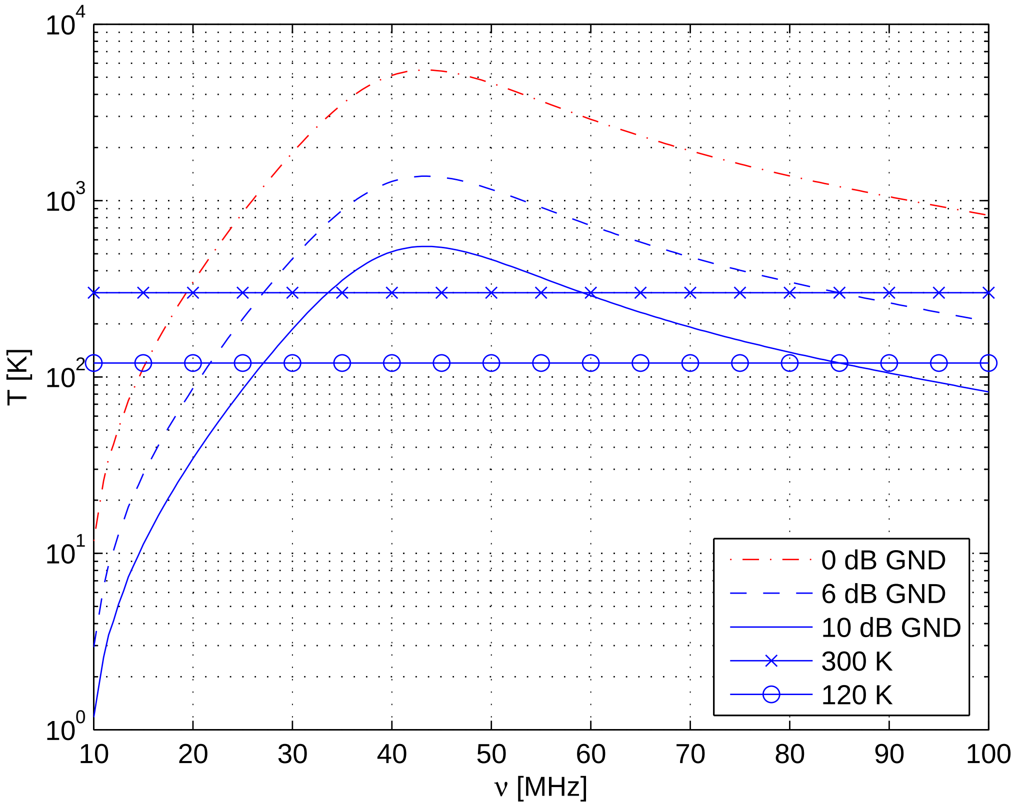

Figure 2: Maximum T_p that yields the indicated degree of Galactic noise domination (GND) assuming the impedance of the "mLWDA" antenna of Memo 32 [4] connected to a 100 Ω active balun input impedance. Also indicated are the noise temperatures (300 K and 120 K) for two existing LWA active balun candidates.

2.3 Requirements for Galactic Noise-Limited Sensitivity

The sensitivity of the LWA is optimized by ensuring T_p is sufficiently small that σ is limited overwhelmingly by T_{sky}. Examination of Equation 5 reveals that this is achieved when

$$T_p \ll \frac{\xi}{\mu} T_{74} \left(\frac{\lambda}{4 \text{ m}}\right)^{2.6} \quad (8)$$

As an example, this is shown graphically in Figure 2 using the ξ characteristic of the "mLWDA" antenna of Memo 32 (see the previous section and Figure 1) and an active balun input impedance of 100 Ω.

2.4 Field of View

The field of view (FOV) of LWA can be defined as the angular area (i.e., deg^2) bounded by the locus of half-power points of a station beam. The half-power beamwidth (HPBW) of a uniformly-excited, equally-spaced linear array of length D is given by [6]

$$\psi(\theta) = \begin{cases} \psi_0 \left(\frac{\lambda}{D}\right) \sec\theta & , \quad \theta \text{ "near" } 0 \\ 2\sqrt{\psi_0 \left(\frac{\lambda}{D}\right)} & , \quad \theta = \frac{\pi}{2} \end{cases} \quad (9)$$

where $\psi_0 = 0.886$. This expression also yields the exact zenith HPBW for a square planar array with sides of length D, and also the exact zenith HPBW for a planar circular array of diameter D if ψ_0 is chosen to be

1.02 [6]. On this basis, we shall assume that the above expression with $\psi_0 = 1.02$ is valid for the beam of a circular LWA station. Although the accuracy of Equation 9 is uncertain for θ far from zenith, we note that the two expressions intersect at θ equal to

$$\theta_c \equiv \arccos\left(\frac{1}{2}\sqrt{\psi_0 \frac{\lambda}{D}}\right) \ . \tag{10}$$

θ_c is greater than 74° for $D = 100$ m and $\psi_0 = 1.02$ (or 0.886), thus we shall assume simply

$$\psi(\theta) = \psi_0 \left(\frac{\lambda}{D}\right) \sec\theta \ . \tag{11}$$

This of course neglects mutual coupling, however we shall leave ψ_0 as a free parameter that might be used to make adjustments to this at a later time if necessary. Also, note that uniform weighting of the aperture is assumed, which results in the narrowest possible beamwidth. For various reasons, we may wish to taper the aperture distribution (for example, to reduce sidelobes), in which case the HPBW will inevitably increase. However, here to we can use ψ_0 as a knob to account for this later. FOV is now given by:

$$\text{FOV} = \frac{\pi}{4}\ \psi^2(\theta)\left(\frac{180°}{\pi}\right)^2 = 2578\ \psi_0^2 \left(\frac{\lambda}{D}\right)^2 \sec^2\theta \ \ [\text{deg}^2] \tag{12}$$

where the leading factor of $\pi/4$ is the ratio of the area of a circle to the area of a square having sides equal in length to the circle's diameter. For later convenience this expression is rewritten:

$$\text{FOV} = 4.12\ \psi_0^2 \left(\frac{\lambda}{4\text{ m}}\right)^2 \left(\frac{D}{100\text{ m}}\right)^{-2} \sec^2\theta \ \ [\text{deg}^2]. \tag{13}$$

3 Number of Sources in the LWA FOV

The calibratibility of LWA depends on the number of sources present in the LWA FOV. Aaron states in [1] that the number of sources per square degree with flux density s or greater in the VLSS and other 74 MHz surveys is

$$N(s) = 1.14\left(\frac{s}{\text{Jy}}\right)^{-1.3} \tag{14}$$

with the caveat that this is only known to be accurate down to about $s = 0.4$ Jy/beam. To extrapolate to other frequencies, it is assumed that the source flux density scales according to the typical spectral index of a low frequency source; i.e., as $\lambda^{-0.7}$. Thus we have:

$$N(s) = 1.14\left(\frac{s}{\text{Jy}}\right)^{-1.3}\left(\frac{\lambda}{4\text{ m}}\right)^{0.91} \quad [\text{deg}^{-2}] \tag{15}$$

The number of sources per square degree with flux density s or greater in the FOV is therefore

$$\begin{aligned} N_{FOV}(s) &= N(s)\cdot\text{FOV} \\ &= 4.70\ \psi_0^2\left(\tfrac{s}{\text{Jy}}\right)^{-1.3}\left(\tfrac{\lambda}{4\text{ m}}\right)^{2.91}\left(\tfrac{D}{100\text{ m}}\right)^{-2}\sec^2\theta \end{aligned} \tag{16}$$

4 Number of Sources Required for "Full Field" Calibration

In [1] Aaron explains that field-based ionospheric calibration with the VLA 74 MHz system in A-Configuration requires 4-6 sources typically and that 10 sources may be desireable. Let this number be N_{cal}^{VLA}. The requirement for LWA can be extrapolated as follows:

$$N_{cal} = N_{cal}^{VLA}\left(\frac{L_B}{36\text{ km}}\right)^2\left(\frac{\text{FOV}}{\text{FOV}_{VLA}}\right)\frac{1}{r_{np}}\left(\frac{\lambda}{4\text{ m}}\right)^2 \tag{17}$$

where:

- L_B is the length of maximum baseline, currently planned to be about 400 km for the goal 52-station system.
- FOV_{VLA} is the FOV of the VLA. This can be obtained from Equation 13 using $\psi_0 = 1.02$, $D = 25$ m, $\theta = 0$, and including an additional factor of 1.13 to account for aperture taper (See [1], Equation 3); the result is 77 deg^2.
- r_{np} is the fraction of detectible sources which appear to be extended (as opposed to being point sources) due to the improved resolution, and thus are not suitable as calibrators. Aaron suggests $r_{np} = 0.5$ in [1]; however Greg Taylor has asserted that from VLBI experience this is not necessarily a limitation, and so it may be OK to set $r_{np} = 1$ for calibration purposes.
- The wavelength dependence accounts for the fact that the number of calibrators required per FOV scales by another factor of λ^2 because the magnitude of ionospheric phase variations is proportional to λ.

Substituting Equation 13 we find:

$$N_{cal} = 0.053\ \psi_0^2 \left(\frac{N_{cal}^{VLA}}{r_{np}}\right) \left(\frac{L_B}{36\text{ km}}\right)^2 \left(\frac{\lambda}{4\text{ m}}\right)^4 \left(\frac{D}{100\text{ m}}\right)^{-2} \sec^2\theta\ . \tag{18}$$

For $N_{cal}^{VLA} = 10$, $r_{np} = 0.5$, $L_B = 360$ km, $\lambda = 4$ m, $D = 100$ m, and $\theta = 0$, this yields $N_{cal} = 111$ sources. This is slightly less than the Memo 52 value of 125 due to the refined estimate of the LWA and VLA FOVs.

5 Number of Stands per Station Required for "Full Field" Calibration

From the considerations above we see that "full field" calibration requires $N_{FOV}(s) \geq N_{cal}$ with $s = r\sigma$, where r is some acceptable level of detection significance; e.g., $r = 5$. Combining Equations 16, 18, and 5 we find that the required number of stands per station is:

$$N_a \geq 174600\ rC \left[\left(\frac{\lambda}{4\text{ m}}\right)^{3.44} + \frac{\mu}{\xi}\frac{T_p}{T_{74}}\left(\frac{\lambda}{4\text{ m}}\right)^{0.84}\right] \left(\frac{A_{e0}(\lambda,\theta,\phi)}{\text{m}^2}\right)^{-1} \left(\frac{N_{cal}^{VLA}}{r_{np}}\right)^{0.77} \left(\frac{L_B}{36\text{ km}}\right)^{1.54} \tag{19}$$

where

$$C \equiv \frac{1}{\eta_s \gamma L_g \sqrt{N_S(N_S-1)N_{pol}\Delta\tau\Delta\nu}}\ . \tag{20}$$

6 A Quick Study of Required Number of Stands per Station

We now consider a quick study of N_a versus ν and θ, facilitated by the antenna model of Equation 7 using gain and impedance values corresponding to the mLWDA antenna, and using the most current information concerning the remaining parameters. Figure 3 shows the results for $r = 5$, $\eta_s = 0.78$, $\gamma = L_g = \mu = 1$, 100Ω active balun input impedance, $N_S = 52$, $N_{pol} = 2$, $\Delta\tau = 6$ s, $\Delta\nu = 8$ MHz, and $L_B = 400$ km. The results are shown for three values of T_p to demonstrate the influence of active balun noise temperature, assuming that the active balun dominates the receiver temperature. Recall that the actual value of γ (coupling effect) might change this (in either direction) by as much as 35%. Also, one should remain wary of the various other model assumptions and implications noted above. Some summary comments on these results are as follows:

- These results are consistent with Aaron's Memo 52 result (74 MHz, $\theta = 0$, $N_{cal}^{VLA} = 10$, $r_{np} = 0.5$). Based on this consideration alone, $N_a = 256$ would still seem to be a good choice.
- N_a is dramatically reduced for the alternative calibratibility assumptions $N_{cal}^{VLA} = 4$ and $r_{np} = 1$. It may be cost-effective for us to improve our understanding of what is really needed here.

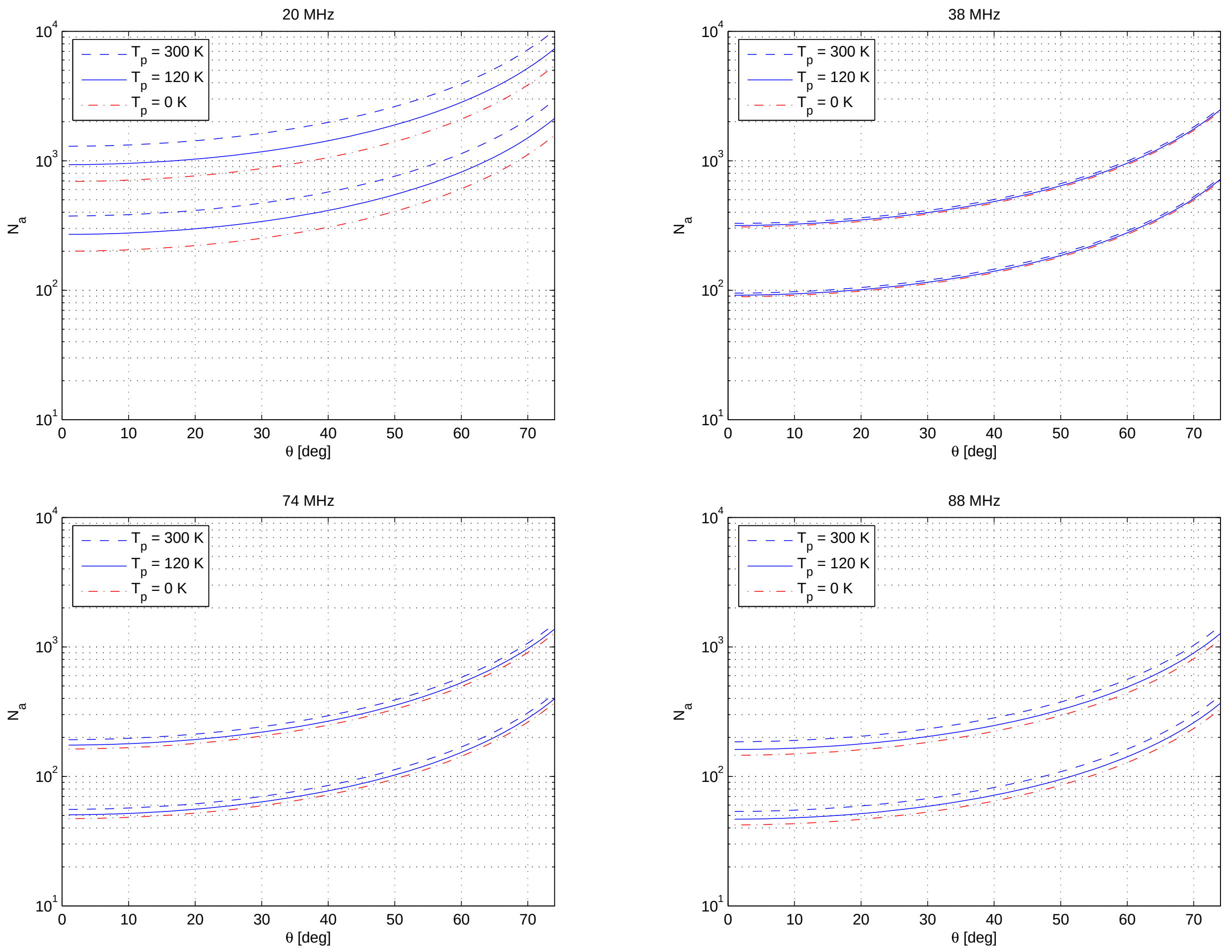

Figure 3: Required stands/station (N_a) for the goal LWA configuration of $N_S = 52$ stations with maximum baseline $L_B = 400$ km. The upper set of curves in each plot assume $N_{cal}^{VLA} = 10$ and $r_{np} = 0.5$ (as in Memo 52) whereas the lower set of curves assume $N_{cal}^{VLA} = 4$ and $r_{np} = 1$ (i.e., are much more optimistic about what is needed for calibratibility).

- The (approximately) $cos^{1.6}\theta$ pattern dependence is a killer, increasing N_a by about an order of magnitude at 74° with respect to zenith pointing. If we want reasonable performance at the Galactic Center, it seems we either need thousands of stands per station or we should seek antennas with better gain at low elevations. In any event, it may be worthwhile to optimize antennas such that some of the "excess gain" at the zenith is shifted to lower elevations.
- 256 stands doesn't seem to cut it at all at 20 MHz. This implies an optimization is warranted between antenna size (bigger antennas yielding better low-frequency performance) and construction cost (bigger antennas are more expensive and harder to install).

Another configuration of interest is $N_S = 16$ and $L_B = 200$ km, which is the smallest system to which "full field" calibration might be applied. The difference from the goal system is an increase in N_a of about 14% which is constant over all cases considered.

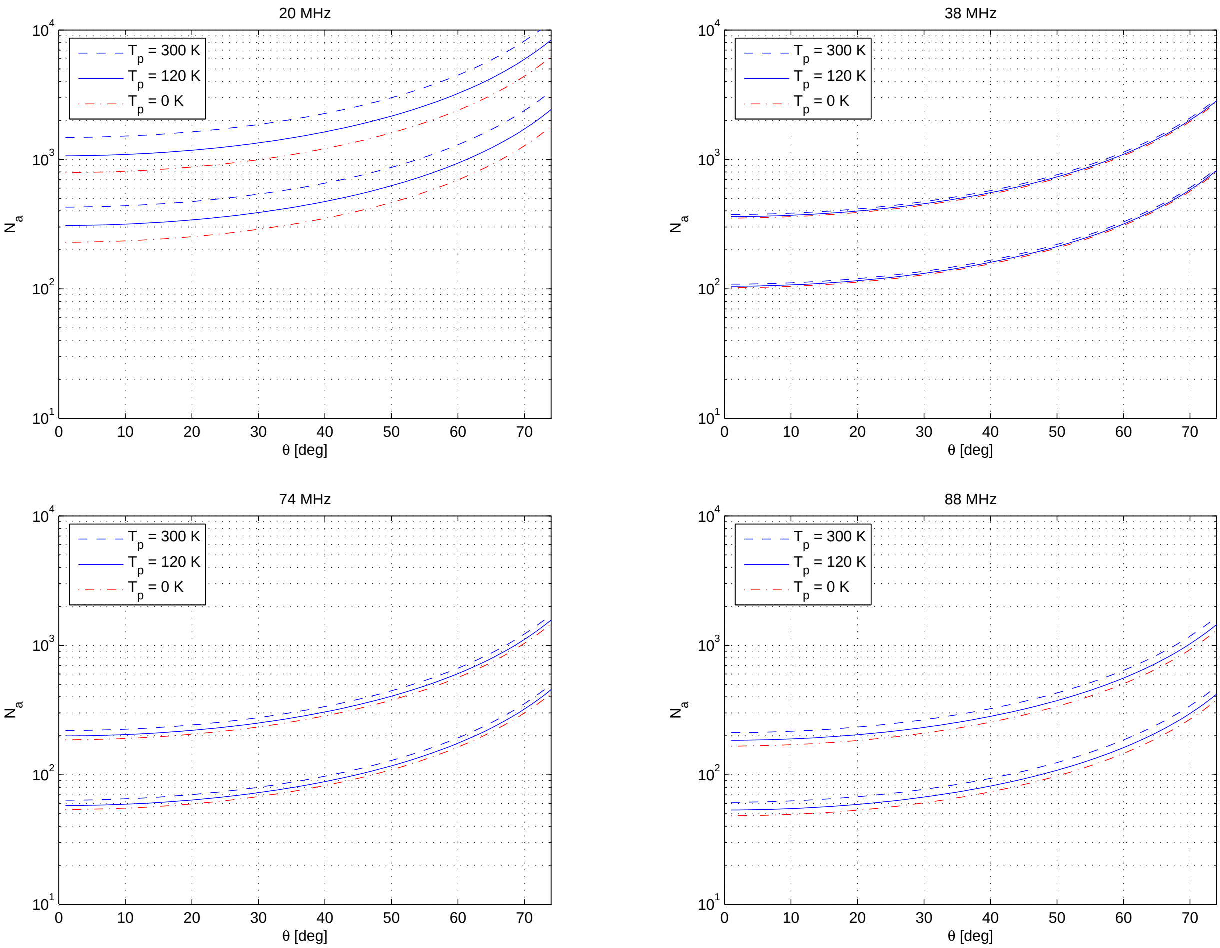

Figure 4: Same as Figure 3, except for an intermediate-size LWA configuration of $N_S = 16$ stations with maximum baseline $L_B = 200$ km. (This results in only a very slight increase which is constant over all cases considered.)

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