Abstract: In this note we examine the zeros of a modal model of flexible structures. It is shown that in some situations the zero patterns change radically with model order. The refinement of the model by the addition of higher order modes can cause the appearance of zeros at low frequencies or in the right half plane. Structures instrumented with piezoelectric actuators and/or fiber optic sensors are also considered.

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I. Introduction

It is well known [1,2] that the zeros of the transfer function of the plant play an important role in the design of the control system for that plant. It is equally important that the design model for the plant capture the zero structure in the frequency band of interest. Recently, several authors have pointed out that the characterization of zeros for flexible structures is particularly important [2-5]. If modal models are used to approximate the infinite dimensional system the zeros of reduced order models of different dimensions may not be well behaved. Therefore, it is difficult to decide which zeros are crucial for control design. Two descriptions of zeros for partial differential equations are in terms of high gain feedback [6], or, in terms of an infinite dimensional transfer function [7]. For many applications, these characterizations can't be applied directly. These problems normally occur in systems with non-collocated sensors and actuators [3].

Recently, there has been an interest in structures with embedded sensors and actuators. These structures often include novel actuators such as piezoelectric actuators [5] and/or novel sensors such as optical fiber sensors [8]. Here we show that these structures have the same properties with respect to modeling the zeros as structures with non-collocated sensors and actuators.

In this paper we examine the relationship between the zeros of finite dimensional modal models of various orders of infinite dimensional systems. Examples are given where the addition of a higher order mode can lead to the appearance of zeros at or below the frequency of the first mode or in the right half plane. We show that the zero patterns of these models can be described in terms of the elements of the input and output matrices of the finite dimensional models. Explicit connections between the sensor and actuator locations, the type of sensor or actuator, the mode shapes, and the zeros are drawn.

II. Model Description

Assume that there exists three sets of numbers:

\[ \{b_k\}_{k=1}^\infty, \{c_k\}_{k=1}^\infty, \text{ and } \{\omega_k\}_{k=1}^\infty \]  

such that \(b_k \neq 0, c_k \neq 0,\) and \(0 \leq \omega_1 < \cdots < \omega_{k-1} < \omega_k\) for all \(k\). From these sets we construct the following dynamic equations:

\[ \ddot{\eta}_i(t) + \Omega_i^2 \eta_i(t) = B_i u(t), \]

\[ y(t) = C_i \eta_i(t), \]  

\[ \tag{2.2} \]
where

\[
\tilde{\eta}_i(t) = \begin{bmatrix} \eta_1(t) \\ \vdots \\ \eta_i(t) \end{bmatrix}, \quad \Omega_i^2 = \begin{bmatrix} \omega_1^2 \\ \vdots \\ \omega_i^2 \end{bmatrix}, \quad B_i = \begin{bmatrix} b_1 \\ \vdots \\ b_i \end{bmatrix}, \\
C_i = \begin{bmatrix} c_1 & \cdots & c_i \end{bmatrix}.
\] (2.3)

Because of the assumptions above, the model in (2.2) is controllable and observable for all \(i\). The transfer function of (2.2) is

\[
\frac{Y(s)}{U(s)} = G_i(s) = \sum_{k=0}^{i} \alpha_k \frac{s^2 + \omega_k^2}{d_i(s)}
\] (2.4)

where the coefficients

\[
\alpha_k = c_k b_k
\] (2.5)

are called the (modal) influence coefficients of this transfer function.

The zeros of \(G_i(s)\) are the roots of \(n_i(s)\) in (2.4). With the assumptions above, there are exactly \(2(i-1)\) zeros which can be grouped in pairs as will be shown below. Denote each pair of zeros as \(z_{ki}\) for \(0 < k < i\).

Notation for the zero locations: 1) the first index is the \(k\)th pair of zeros, 2) the second index is the number of retained coefficients (modes).

It is clear from (2.4) that if one more mode is added to the model (2.2), then the poles of the transfer function \(G_i(s)\) will coincide with \(2i\) poles of \(G_{i+1}(s)\). In this note we study the behavior of the zeros of \(G_i(s)\) as the order of the transfer functions increase.

Clearly we are motivated by so called modal models of flexible structures. The following example is typical.

**Example 2.1.** Consider the Euler-Bernoulli model of a uniform cantilever beam of length \(L\)

\[
EI \frac{\partial^4 w(z,t)}{\partial z^4} + \rho_a \frac{\partial^2 w(z,t)}{\partial t^2} = \delta(z-z_a)u(t)
\] (2.6)

where \(w(z,t)\) is the displacement of the beam at point \(z\) at time \(t\) due to a point force actuator at \(z_a\). Here \(EI\) is the flexural rigidity and \(\rho_a\) is the linear mass density. The boundary conditions are
\[ w(0,t) = \frac{\partial w(0,t)}{\partial z} = \frac{\partial^2 w(L,t)}{\partial z^2} = \frac{\partial^3 w(L,t)}{\partial z^3} = 0. \quad (2.7) \]

The modal model (2.2) can be constructed directly from the partial differential equation (2.6) using separation of variables. The displacement of the beam, \( w(z,t) \) is written as

\[ w(z,t) = \sum_{k=0}^{\infty} \psi_k(z) \eta_k(t) \quad (2.8) \]

where the modeshapes, \( \psi_k(z) \), can be calculated from (2.6) and (2.7). If the infinite sum in (2.8) is truncated to \( i \) terms then the modal amplitudes can be calculated from the model in (2.2).

We assume that a force actuator is located at the tip of the beam, and a displacement sensor is located at \( z_s = 0.45L \) from the root of the beam. Then the input and output matrices in (2.3) are defined by

\[ b_k = \psi_k(L), \quad \text{and} \quad c_k = \psi_k(0.45L). \quad (2.9) \]

For these sensor and actuator locations we compute the zeros for each transfer function as the index \( i \) is increased from \( i = 2 \) to 5. The zeros of the transfer functions \( G_i(s) \) are shown in Table 2.1 along with the modal frequencies. These zeros appear in pairs and they are symmetric about the origin. This pattern is typical of the zero behavior for modal models of structures with non-collocated sensors and actuators.

| Table 2.1. Zeros and Modal Frequencies of the Cantilevered Beam |
|--------------------------|------------------|------------------|------------------|------------------|------------------|
| \( i \) | 1 | 2 | 3 | 4 | 5 |
| \( \omega_i \) rad/s | 0.1570 | 0.9839 | 2.7549 | 5.3986 | 8.9243 |
| zeros | \( \pm 0.7684 \) | \( \pm j4.4724 \) | \( \pm 8.2820 \) | \( \pm 3.7910 \) | \( \pm j3.7673 \) |
| | | | | | \( \pm 5.4688 \) |
| | | | | | \( \pm j3.4277 \) |
| | | | | | \( \pm 0.8737 \) |
| | | | | | \( \pm 0.8595 \) |

Note the widely erratic behavior of the zeros of smallest magnitude. Also note that the addition of higher order modes can cause the appearance of zeros in the proximity of the lowest order modes. In the next section we offer an explanation of this behavior.
III. Zero Analysis Via (Modal) Influence Coefficients

In this section we present one explanation for the behavior of the zeros of the transfer functions in (2.4) above. For \( i > 1 \), we have

\[
G_{i+1}(s) = \frac{n_i(s)}{d_i(s)} + \frac{\alpha_{i+1}}{s^2 + \omega_{i+1}^2} = \frac{n_i(s)[s^2 + \omega_{i+1}^2] + \alpha_{i+1} d_i(s)}{d_i(s)[s^2 + \omega_{i+1}^2]} \tag{3.1}
\]

where

\[
d_i(s) = \prod_{k=1}^{i} (s^2 + \omega_k^2). \tag{3.2}
\]

Then the zeros of \( G_{i+1}(s) \) are the roots of the numerator in (3.1). The location of these zeros can be studied by examining the root loci of the polynomials

\[
n_{i+1}(s; k) = n_i(s)[s^2 + \omega_{i+1}^2] + kd_i(s) \tag{3.3}
\]

for \( 0 \leq k \leq \alpha_i + 1 \) if \( 0 \leq \alpha_i + 1 \), or for \( \alpha_i + 1 \leq k \leq 0 \) if \( \alpha_i + 1 \leq 0 \). By comparing the qualitative behavior of the root loci of (3.3) as \( i \) increases, insight is gained into the behavior of the zeros.

**Example 3.1.** Assume that all of the (modal) influence coefficients (2.5) are positive. For this system, the root loci which describes the zero behavior for \( i = 2 \) is shown in Fig. 3.1a and for \( i = 3 \) in Fig. 3.1b. For \( i = 2 \), (3.3) becomes

\[
n_2(s; k) = \alpha_1 (s^2 + \omega_2^2) + k(s^2 + \omega_1^2). \tag{3.4}
\]

As \( k \) varies from 0 to \( \infty \), the roots of (3.4) vary from the "poles" \( \pm j\omega_2 \), marked in Fig. 3.1a by an "x," to the "zeros," \( \pm j\omega_1 \), marked with a "O." The roots of \( n_2(s) \), the zeros of \( G_2(s) \), are found by setting \( k = \alpha_2 \). These roots are marked by "□." We see that \( \omega_1 < \left| z_{12} \right| < \omega_2 \) because the influence coefficients, \( \alpha_1 \) and \( \alpha_2 \), are positive.

For \( i = 3 \), we have

\[
n_3(s; k) = n_2(s)(s^2 + \omega_3^2) + k(s^2 + \omega_1^2)(s^2 + \omega_2^2). \tag{3.5}
\]

In Fig. 3.3b, the poles, marked by "x," are \( \pm j\omega_3 \) and the zeros of \( G_2(s) \). The zeros, marked by "O," are \( \pm j\omega_1 \) and \( \pm j\omega_2 \). The roots of \( n_3(s) \), the zeros of \( G_3(s) \), are found by setting \( k = \alpha_3 \). These roots are marked by "□" in Fig. 3.1b.
We see that $\omega_1 < |z_{13}| < |z_{12}| < \omega_2 < |z_{23}| < \omega_3$ because the (modal) influence coefficient are all positive. Extrapolating this argument we see that the poles and zeros will interlace on the imaginary axis for systems in which the (modal) influence coefficient are all positive.

**Example 3.2.** Consider the system (2.2) in which the output equation is given by

$$y_v(t) = C_i \ddot{\eta}_i(t). \quad (3.6)$$

Then the transfer function is given by

$$\frac{Y_v(s)}{U(s)} = sG_i(s) = \frac{sn_i(s)}{d_i(s)}. \quad (3.7)$$

If the measurement (3.6) is a velocity measurement collocated with a force input, then $b_{ki} = c_{ki}$ for all $k$ and the influence coefficients are all positive. From Example 3.1, the poles and zeros will interlace on the imaginary axis, as is well known [9] since this transfer function is positive real.

**Example 3.3.** Consider again Example 3.1 and assume that $0 < \alpha_i$ for $i=1,2$ but $0 > \alpha_3$. From (3.4) it can be seen that
\[ n_2(s) = (\alpha_1 + \alpha_2) \hat{n}_2(s) \]  

(3.8)

where \( \hat{n}_2(s) \) is a monic polynomial. If \( 0 < |\alpha_3| < (\alpha_1 + \alpha_2) \) then \( \omega_1 < |z_{12}| < |z_{13}| < \omega_2 \) but \( |z_{23}| > \omega_3 \) and the zeros will be in a complex pair on the imaginary axis as shown in Fig. 3.2a.

If \( (\alpha_1 + \alpha_2) < |\alpha_3| \) then the zeros will lie on the loci which corresponds to the real axis, in effect having passed through the point of infinity when \( |\alpha_3| = (\alpha_1 + \alpha_2) \). This situation is illustrated in Fig. 3.2b. Recall that the root locus generally exhibits extreme sensitivity to changes in the gain when \( k \approx (\alpha_1 + \alpha_2) \).

This example shows how the addition of a high frequency mode to the finite dimensional model can generate a pair of zeros at low frequencies. See \( i = 4 \) in Table 2.1.

**Example 3.4.** Consider again Example 3.1 and assume that \( \alpha_2 < 0 \) but \( \alpha_i > 0 \) for \( i=1,3 \). The root locus of the zeros for \( i = 2 \) is shown in Fig. 3.3. The zeros are marked by a "\( \square \)" for \( \alpha_1 < |\alpha_2| \). We consider two cases.

Case 1. First consider the third order model when \( |\alpha_1 + \alpha_2| > \alpha_3 \). The zeros, marked in Fig. 3.4, are on the imaginary axis and on the real axis.
Here the zeros on the real axis for $i = 3$ are close to the zeros on the real axis for $i = 2$. A pair of complex zeros have appeared on the imaginary axis below the poles added by the inclusion of the third mode.

Case 2. Next suppose that $|\alpha_1 + \alpha_2| < \alpha_3$. The zeros of the third order model shown in Fig. 3.5 are marked by a "□" for one possible $\alpha_3$. The three mode model has four zeros which are symmetric with respect to the real and imaginary axis. The locations of these zeros are sensitive to the influence coefficients, $\alpha_i$, for $i = 1, 2, 3$.

This example shows that when the influence coefficients have alternating signs, very complex zero patterns can exist, and these patterns can fluctuate wildly with the model order.
IV. Cantilevered Beam

In this section we consider the cantilevered beam in Example 2.1. This beam is shown in Fig. 4.1 along with the first three modeshapes.

We assume that the beam is instrumented with a point force actuator at the tip. Then we examine the zero patterns as the sensor location is varied. The elements of the output matrix, \( c_{ki} \), are just the mode shapes evaluated at the sensor location.

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When the sensor is located at the tip of the beam, the elements of the output matrix are also all of the same sign as the corresponding element in the input matrix. Hence, the influence coefficients of this system are all positive and the zeros will interlace with the poles as in Example 3.1.

As the displacement sensor is moved toward the root of the beam it will cross the node of the third mode. At that point the third mode's influence coefficient will change sign, the corresponding elements of the input and output being of opposite sign. This situation is described by Example 3.3. The first pairs of zeros will be interlaced with the first two poles. The second pair of zeros, however, will be at a higher frequency than the third mode or they will be a symmetric pair on the real axis.

As the sensor is moved closer and closer to the root of the beam it crosses the nodes of both the second and third nodes. In this region the first and third modal influence coefficients are positive but the second modal influence coefficient is negative. This case is discussed in Example 3.4. For the two sensor locations shown in Fig. 4.1 the relative values of $|\alpha_2|$ and $(\alpha_1 + \alpha_3)$ differ as discussed in Example 3.4, Case 1 and Case 2, respectively. The corresponding sensor locations are identified in Fig. 4.1. In this situation the zeros form complex patterns that are sensitive to small changes in the influence coefficients. Hence, small shifts in the sensor location may have dramatic effect on the zeros of the transfer function.

**Example 4.1.** Suppose that the displacement sensor is fixed at the point 0.95L, that is just to the right of the node corresponding to the third mode in Fig. 4.1. If the system parameters change, say, by the addition of a mass at the tip of the beam or through a loss of stiffness somewhere along the beam, then it is possible that the node of the third mode may shift to the other side of the sensor. This shift in the node will cause the sign of the residue to change and the pole/zero interlacing pattern will change as in Example 3.3. This phenomenon is called "zero flipping" [3]. This phenomenon is one of the reasons systems with non-collocated sensors and actuators have robustness problems.

**Example 4.2.** Consider again the cantilevered beam of Example 2.1. Assume that the beam is instrumented with a piezoelectric bending motor at the root of the beam and a modal domain optical fiber sensor attached along the length of the beam [8]. The modal domain optical fiber sensor is novel in that it responds to the strain in the beam over a significant gauge length. We call such sensors distributed-effect sensors.

It can be shown that the elements of the input matrix, $B$, are given by

$$b_{ki} = b_0\left[\psi'_k(p_2) - \psi'_k(p_1)\right] \quad (4.1)$$
where $b_0 > 0$ and $p_2 > p_1$ are the endpoints of the piezoelectric patch. If the optical fiber sensor is attached from the root to the point $z = z_s$ along the beam, the output equation in (2.2) becomes

$$c_{ki} = c_0 \left[ \psi_k(z_s) - \psi_k'(0) \right] = c_0 \psi_k(z_s) \tag{4.2}$$

where $c_0 > 0$.

Consider again the mode shapes shown in Fig. 4.1. Fig. 4.1 shows that for a small piezoelectric patch near the root of the beam, the elements in the input matrix (4.1) alternate in sign for the low order modes. The elements in the output matrix, (4.2), however, are all positive. It follows that the (modal) influence coefficients are both positive and negative. Consequently, a complex zero pattern would be expected. The zero pattern for a sixth order modal model is shown in Fig. 4.2.

![Figure 4.2. Zeros and Poles of a Cantilevered Beam with a Modal Domain Fiber Optic Sensor and Piezoelectric Actuator.](image)

The zeros close to the origin are due to the low order modes. Note that the zeros due to the higher order modes appear in the right and left hand plane as well as interlaced with the poles on the imaginary axis.

**Remark 4.3:** Suppose that the endpoint of the optical fiber sensor is not located at the end of the beam, but is located somewhere along the length of the beam, say, $z_s = 0.85L$. A change in the modeshapes could cause the third modal influence coefficient to change sign, leading to a change in the zero pattern. This analysis suggests that a distributed-effect sensor does not posses any inherent qualities which would lead to better robustness over a point sensor.
V. Conclusions

In this note we have examined the behavior of zeros of modal models which are commonly used to model flexible structures. We have shown that for structures with non-collocated sensors and actuators and systems with distributed-effect sensors and actuators the zeros can vary widely depending on the number of modes retained in the finite dimensional model.

VI. References


