ECE 5984: Power Distribution System Analysis

Lecture 12: DistFlow and LinDistFlow
References


Outline

1. Branch flow model (BFM)
2. DistFlow model
3. DistFlow model for power flow
4. DistFlow model for optimal power flow
5. LinDistFlow model for approximate analysis
Single-phase radial feeder

- **Single-phase** and **radial** feeder represented by tree graph

\[ \mathcal{G} = (\mathcal{N}, \mathcal{E}) \text{ with } |\mathcal{N}| = N + 1 \text{ and } |\mathcal{E}| = L = N \]

**breadth-first** (vs. **depth-first**) numbering

- Branch-bus incidence matrix \( \tilde{A} = [a_0 \ A] \)

\[
\tilde{A}1_{N+1} = 0 \Rightarrow \quad a_0 + A1_N = 0 \\
1_N = -A^{-1}a_0
\]

**reduced branch-bus incidence matrix**

\[
A = \begin{bmatrix}
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix}
\]

\[
F = A^{-1} = \begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix}
\]
Branch flow model (BFM)

\[ V_{\pi n} - V_n = z_n I_n \]
\[ S_n = V_{\pi n} I_n^* \]
\[ S_n - z_n |I_n|^2 + s_n = \sum_{k: n \rightarrow k} S_k \]

Line \( n \) feeding bus \( n \) from its parent bus \( \pi_n \)

- Branch flow equations on \( x(s) := (S, I, V, s_0) \)

- Boundary conditions?

- Given \( s \), solve 2L+N+1 equations in 2L+N+1 complex unknowns [3]

- Equivalent with typical bus injection model (BIM); a.k.a. power flow equations
Relaxed branch flow model

- Relaxed BFM on $y(s) := (S, \ell, v, p_0, q_0)$

\[
\sum_{k: n \rightarrow k} P_k = p_n + P_n - r_n \ell_n
\]

\[
\sum_{k: n \rightarrow k} Q_k = q_n + Q_n - x_n \ell_n
\]

\[
v_n = v_{\pi n} - 2r_nP_n - 2x_nQ_n + (r_n^2 + x_n^2)\ell_n
\]

\[
\ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi n}}
\]

- Boundary conditions?
- Current mags. can be eliminated; equations remain nonlinear
- Given $s$, solve $2(L+N+1)$ equations in $3L+N+2$ real unknowns [1]-[2]
- In radial grids, we get $4N+2$ equations in $4N+2$ real unknowns
- Unique solution for practical networks with $v_0 \simeq 1$ and small $\{r_n, x_n\}$

$v_n := |V_n|^2$ squared voltage mag.

$\ell_n := |I_n|^2$ squared current mag.

*current and voltage phases have been dropped!*
Recovering phases

- After the relaxed branch flow equations have been solved [3]

- Recover voltage phases

\[
V_{\pi_n} - V_n = z_n I_n \Rightarrow \\
V_{\pi_n}^* - V_n^* = z_n^* I_n^* \Rightarrow \\
V_{\pi_n} V_n^* = v_{\pi_n} - z_n^* S_n \]

\[
\theta_n - \theta_{\pi_n} = \angle(v_{\pi_n} - z_n^* S_n) \quad \text{linear system can be inverted only when } L=N
\]

- Recover current phasors \[ I_n = \left( \frac{S_n}{V_{\pi_n}} \right)^* \]
Backward relaxed BFM

- Previously presented the forward version of the BFM equations

- There is also a backward counterpart

\[
\begin{align*}
P_n &= \sum_{k: \ n \rightarrow k} (P_k - r_n\ell_n) + p_n \\
Q_n &= \sum_{k: \ n \rightarrow k} (Q_k - x_n\ell_n) + q_n \\
v_{\pi_n} &= v_n - 2r_nP_n - 2x_nQ_n + (r_n^2 + x_n^2)\ell_n \\
\ell_n &= \frac{P_n^2 + Q_n^2}{v_n}
\end{align*}
\]

\(S_n\) is now the power sent from child to parent node
Backward/forward solver for BFM model

**Backward sweep**

\[
P_n = \sum_{k: n \rightarrow k} (P_k - r_n l_n) + p_n
\]

\[
Q_n = \sum_{k: n \rightarrow k} (Q_k - x_n l_n) + q_n
\]

\[
v_{\pi_n} = v_n - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) l_n
\]

\[
l_n = \frac{P_n^2 + Q_n^2}{v_n}
\]

**Forward sweep**

\[
\sum_{k: n \rightarrow k} P_k = p_n + P_n - r_n l_n
\]

\[
\sum_{k: n \rightarrow k} Q_k = q_n + Q_n - x_n l_n
\]

\[
v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) l_n
\]

\[
l_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}
\]

- Works only for feeders without laterals
Backward/forward solver for BFM model (cont’d)

**Backward sweep**

\[
\begin{align*}
P_n &= \sum_{k: n \rightarrow k} (P_k - r_n \ell_n) + p_n \\
Q_n &= \sum_{k: n \rightarrow k} (Q_k - x_n \ell_n) + q_n \\
v_{\pi_n} &= v_n - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
\ell_n &= \frac{P_n^2 + Q_n^2}{v_n}
\end{align*}
\]

**Forward sweep**

\[
\begin{align*}
\sum_{k: n \rightarrow k} P_k &= p_n + \ell_n - r_n \ell_n \\
\sum_{k: n \rightarrow k} Q_k &= q_n + \ell_n - x_n \ell_n \\
v_n &= v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
\ell_n &= \frac{P_n^2 + Q_n^2}{v_{\pi_n}}
\end{align*}
\]

• Combine two versions for general feeders with laterals
Linearized distribution flow (LinDistFlow)

- Approximate model to overcome the complexity of quadratic equations [1]-[2]
- Derived from forward DistFlow model upon dropping terms related to losses

**DistFlow (forward form)**

\[
\begin{align*}
\sum_{k: \ n \rightarrow k} P_k &= p_n + P_n - r_n \ell_n \\
\sum_{k: \ n \rightarrow k} Q_k &= q_n + Q_n - x_n \ell_n \\
v_n &= v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
\ell_n &= \frac{P_n^2 + Q_n^2}{v_{\pi_n}}
\end{align*}
\]

**LinDistFlow**

\[
\begin{align*}
\sum_{k: \ n \rightarrow k} P_k &\approx p_n + P_n \\
\sum_{k: \ n \rightarrow k} Q_k &\approx q_n + Q_n \\
v_n &\approx v_{\pi_n} - 2r_n P_n - 2x_n Q_n
\end{align*}
\]

Voltage drop and line power flows are approximately **linearly** related to power injections
Comparison to Lecture 3

• Drop in squared voltage magnitudes from LDF
  \[ v_{\pi_n} - v_n \simeq 2r_n P_n + 2x_n Q_n \]

• Drop in voltage magnitudes from chapter 3
  \[ |V_{\pi_n}| - |V_n| \simeq \text{Re}\{z_n I_n\} \]

• How are these two approximations related?

• Consider analysis in per unit wlog
  \[
  v_n = |V_n|^2 \simeq |V_0|^2 + 2|V_0|(|V_n| - |V_0|) \\
  = 1 + 2(|V_n| - 1) = 2|V_n| - 1
  \]

  \[ v_{\pi_n} - v_n \simeq 2(|V_{\pi_n}| - |V_n|) \]

  \[ r_n P_n + x_n Q_n = \text{Re}\{z_n S_{n}^*\} = \text{Re}\{z_n I_n V_{\pi_n}^*\} \simeq \text{Re}\{z_n I_n\} \]

• Equivalent useful approximation
  \[ |V_{\pi_n}| - |V_n| \simeq r_n P_n + x_n Q_n \]
LDF in compact form

- Express LDF in matrix-vector notation

\[ \sum_{k: n \to k} \hat{P}_k \simeq p_n + \hat{P}_n \]
\[ \sum_{k: n \to k} \hat{Q}_k \simeq q_n + \hat{Q}_n \]
\[ \hat{v}_n \simeq \hat{v}_{\pi n} - 2r_n \hat{P}_n - 2x_n \hat{Q}_n \]
\[ p = A^T \hat{P} \]
\[ q = A^T \hat{Q} \]
\[ A \hat{v} + v_0 a_0 = 2D_r \hat{P} + 2D_x \hat{Q} \]
\[ \hat{v} = v_0 1 + 2FD_r F^T p + 2FD_x F^T q \]
\[ = v_0 1 + 2R p + 2X q \]

- Matrices \((R, X)\) are symmetric positive definite and have positive entries

<table>
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<tr>
<th>Feeder</th>
<th>(\alpha_{\text{min}})</th>
<th>(\alpha_{\text{max}})</th>
<th>mean</th>
<th>std</th>
<th>median</th>
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<td>2.02</td>
<td>0.74</td>
<td>0.38</td>
<td>0.97</td>
</tr>
</tbody>
</table>

- Both matrices are almost equally important
IEEE 13-bus feeder

- Assume transposed lines; average (off)-diagonal entries; take positive-sequence impedance

\[ \hat{v} = v_0 1 + 2R_p + 2X_q \]

- To find entry \( R_{mn} \) connect buses \( n \) and \( m \) to the substation, and add the resistances of the common lines, e.g., \( R_{10,12} = r_{01} + r_{15} \)
Southern California Edison 47-bus feeder

Matrix X
IEEE 123-bus feeder

Matrix $X$
LDF approximation error

• Express DistFlow in matrix-vector notation

\[
\sum_{k: \text{n} \rightarrow \text{k}} P_k = p_n + P_n - r_n \ell_n
\]

\[
\sum_{k: \text{n} \rightarrow \text{k}} Q_k = q_n + Q_n - x_n \ell_n
\]

\[
v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) \ell_n
\]

\[
p = A^T P + D_r \ell
\]

\[
q = A^T Q + D_x \ell
\]

\[
A v + a_0 1 = 2D_r P + 2D_x Q - (D_r^2 + D_x^2) \ell
\]

• LDF gives an overestimator for squared voltage magnitudes

\[
v = \hat{v} + F D_r \left[ -I - 2F^T \right] D_r \ell + F D_x \left[ -I - 2F^T \right] D_x \ell \leq \hat{v}
\]

\[
\leq 0
\]

• LDF gives an underestimator for line flows

\[
P = F^T p - F^T D_r \ell \geq \hat{P}
\]

• Approximation accuracy depends on loading conditions
Linearized power flow models

• Recall linearized or so-termed DC power flow model in transmission systems $p = B\theta$

• It has been derived under three approximations:
  1. Voltage magnitudes close to unity $|V_n| = 1 + \epsilon_n$
  2. Voltage angle differences across lines close to zero
  3. Ignoring line resistances and shunt elements

• Repeat the same analysis for a meshed grid without the third assumption [6]

$$S_{nm} \sim \frac{(\epsilon_n - \epsilon_m) - j(\theta_n - \theta_m)}{z_{nm}^*} = [(\epsilon_n - \epsilon_m) - j(\theta_n - \theta_m)](g_{nm} + jb_{nm})$$

• Stacking line power flows

$$P = D_gA\epsilon + D_bA\theta$$
$$Q = D_bA\epsilon - D_gA\theta$$

• Converting to power injections

$\text{compare to 'DC' model for transmission grids}$

$$p = A^TP = G\epsilon + B\theta \quad G := A^TD_gA$$
$$q = A^TQ = B\epsilon - G\theta \quad B := A^TD_bA$$
Linearized power flow models (cont’d)

• Solve equations wrt voltage magnitudes and angles

\[ \varepsilon = (G + BG^{-1}B)^{-1} p + (B + GB^{-1}G)^{-1} q \]
\[ \theta = (B + GB^{-1}G)^{-1} p - (G + BG^{-1}B)^{-1} q \]

• Approximation holds for meshed grids

• For radial grids (square and invertible \( A \)), equations simplify to

\[ \varepsilon = Rp + Xq \]
\[ \theta = Xp - Rq \]

• Compare to LDF; linear approximation for voltage angles too

• Linearization conducted at flat voltage profile

• Another reference state can be used; but \((R,X,B,G)\) will depend on that state
Power flow via convex relaxation

• Instead of the BF solver, solve the PF problem as a minimization [3]-[4]

\[
\begin{align*}
\min & \quad \sum_{n=1}^{N} r_n \ell_n \\
\text{over} & \quad P, Q, v, \ell, p_0, q_0 \\
\text{s.t.} & \quad \sum_{k: n\rightarrow k} P_k = p_n + P_n - r_n \ell_n \\
& \quad \sum_{k: n\rightarrow k} Q_k = q_n + Q_n - x_n \ell_n \\
& \quad v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
& \quad \ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \quad \rightarrow \quad \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n
\end{align*}
\]

• Non-convex constraint relaxed to second-order cone constraints (SOC)

\[
\left\| \begin{bmatrix} 2P_n \\ 2Q_n \\ \ell_n - v_{\pi_n} \end{bmatrix} \right\|_2 \leq \ell_n + v_{\pi_n}
\]

• It can be solved efficiently as a second-order cone program (SOCP)

• Oftentimes, the relaxation is exact: SOC are satisfied with equality
Optimal power flow via convex relaxation

• OPF has to be solved to perform any meaningful grid optimization task
  1. power loss minimization
  2. voltage regulation
  3. conservation voltage reduction
  4. demand response
  5. electric vehicle charging
  6. optimal coordination of energy storage

• Power injections $s$ become *control variables* rather than *inelastic load*

• Optimally control devices while satisfying the PF equations and network constraints
Optimal power flow via convex relaxation

- Solving OPF in single-phase radial grids through via an SOCP [3]

\[
\begin{align*}
\text{min} & \quad \sum_{n=1}^{N} r_n \ell_n + \sum_{n=1}^{N} c_n p_n^g + \sum_{n=1}^{N} \alpha_n v_n \\
\text{over} & \quad P, Q, V, \ell, p_0, q_0, s \\
\text{s.t.} & \quad \sum_{k: n \to k} P_k = p_n + P_n - r_n \ell_n \\
& \quad \sum_{k: n \to k} Q_k = q_n + Q_n - x_n \ell_n \\
& \quad v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
& \quad \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n \\
& \quad p = p^g - p^c \quad \text{injection constraints} \\
& \quad q = q^g - q^c \quad \text{constraints} \\
& \quad p_n^g \leq p_n \leq \bar{p}_n^g, \ \forall n \\
& \quad q_n^g \leq q_n \leq \bar{q}_n^g, \ \forall n \\
& \quad (p_n^g)^2 + (q_n^g)^2 \leq s_n^g, \ \forall n \\
& \quad v_n \leq v \leq \bar{v}, \ \forall n \\
& \quad \ell_n \leq \ell, \ \forall n \quad \text{network constraints}
\end{align*}
\]

- Oftentimes, the relaxation is exact: SOCs are satisfied with equality
Exactness under load oversatisfaction

**Theorem ([3]):** If power injections are unbounded below, the relaxation is exact

- Assume problem has been solved, but SOC for line $n$ is inexact $P_n^2 + Q_n^2 < \ell_n v_{\pi_n}$

  $v'_{\pi_n} = v_{\pi_n}$
  $S'_n = S_n - \frac{z_n \epsilon}{2}$
  $\ell'_{n} = \ell_n - \epsilon$
  $s'_{\pi_n} = s_{\pi_n} - \frac{z_n \epsilon}{2}$
  $s'_n = s_n - \frac{z_n \epsilon}{2}$

- Given current solution $(S, s, v, \ell, s_0)$, construct another point $(S', s', v', \ell', s'_0)$ by changing only the quantities related to line $n$ as shown above

- Show that new point is feasible; satisfies SOC with equality; and yields lower cost!

$$
\sum_{k: \text{n} \rightarrow k} P'_k = p'_n + P'_n - r_n \ell'_n
$$

$$
\sum_{k: \text{n} \rightarrow k} Q'_k = q'_n + Q'_n - x_n \ell'_n
$$

$$
v_n = v_{\pi_n} - 2r_n P'_n - 2x_n Q'_n + (r_n^2 + x_n^2) \ell'_n
$$

$$
\ell'_n = \frac{(P'_n)^2 + (Q'_n)^2}{v_{\pi_n}}
$$
Exactness of SOCP convex relaxation

• The exactness of the SOCP convex relaxation for OPF in radial grids has been studied extensively [6]

• Different sets of sufficient conditions have been derived:
  ▪ no reverse power flows
  ▪ identical $r/x$ ratios for all lines
  ▪ $r/x$ increase downstream and there are no reverse active power flows
  ▪ $r/x$ decrease downstream and there are no reverse reactive power flows

• If the SOCP is exact, the minimizer is unique

• To make BFM exact for meshed grids, add phase shifters to implement angle differences [3]

• Otherwise, one can use a semidefinite program relaxation based on the bus injection model (BIM) [4]

• How do these schemes extend to multiphase grids? [7]
Multiphase branch flow model

\[ V_{\pi n} - V_n = z_n I_n \]
\[ S_n = V_{\pi n} I_n^* \]
\[ S_n - z_n |I_n|^2 + s_n = \sum_{k: n\rightarrow k} S_k \]

\[ V_{\pi n} - v_n = Z_n i_n \]
\[ S_n = v_{\pi n} i_n^H \]
\[ \text{matrix variable?} \]
\[ \text{from scalars to vectors and matrices} \]

\[ \sum_{k: n\rightarrow k} \text{dg} (S_k) \]

- Power received at node \( n \)
  \[ \text{dg} (v_n i_n^H) = \text{dg} \left[ (v_{\pi n} - Z_n i_n) i_n^H \right] \]

- Actual power sent from parent bus \( \sigma_n = \text{dg}(S_n) \)
• ‘Square’ (multiply by conjugate transpose) the voltage drop equation

\[ v_n = v_{\pi n} - Z_n i_n \]

\[ v_n v_n^H = v_{\pi n} v_{\pi n}^H + Z_n i_n i_n^H Z_n^H - v_{\pi n} i_n^H Z_n^H - Z_n i_n v_{\pi n}^H \]

• Define ‘squared’ voltages and currents

\[ V_n = v_n v_n^H \quad L_n = i_n i_n^H \]

• Express ‘squared’ voltage drop as

\[ V_n = V_{\pi n} + Z_n L_n Z_n^H - S_n Z_n^H - Z_n S_n^H \]

• Linear equation; but complexity is hidden under ‘squared’ variables \((V_n, L_n, S_n)\)
Relaxed multiphase BFM (cont’d)

- In single-phase grids, \( S_n = V_{\pi_n} I_n^* \)
  
  \[
  |S_n|^2 = v_{\pi_n} \ell_n \leq v_{\pi_n} \ell_n
  \]

- Relaxation can be also written
  \[
  \begin{bmatrix}
  v_{\pi_n} & S_n \\
  S_n^* & \ell_n
  \end{bmatrix}
  =
  \begin{bmatrix}
  V_{\pi_n} \\
  I_n
  \end{bmatrix}
  \begin{bmatrix}
  V_{\pi_n} \\
  I_n
  \end{bmatrix}^H \succeq 0 \text{ and rank-1}
  \]

- In multi-phase grids, the relaxation becomes
  \[
  \begin{bmatrix}
  V_{\pi_n} & S_n \\
  S_n^* & \ell_n
  \end{bmatrix}
  =
  \begin{bmatrix}
  v_{\pi_n} \\
  i_n
  \end{bmatrix}
  \begin{bmatrix}
  v_{\pi_n} \\
  i_n
  \end{bmatrix}^H \succeq 0 \text{ and rank-1}
  \]

- Semidefinite program (SDP) constraint captures all quadratic relationships
OPF with multiphase BFM

\[
\begin{align*}
\min & \quad \text{losses and/or CVR and/or generation cost} \\
\text{over} & \quad \{S_n, s_n, V_n, L_n\}_n \\
\text{s.t.} & \quad \text{relaxed BFM equations} \\
& \quad \begin{bmatrix} V_{\pi_n} & S_n \\ S_n^* & L_n \end{bmatrix} \succeq 0 \\
p & = p^g - p^c \\
q & = q^g - q^c \\
\begin{align*}
p^g_{n,\phi} & \leq p_{n,\phi} \leq \bar{p}^g_{n,\phi}, \forall n, \phi \\
q^g_{n,\phi} & \leq q_{n,\phi} \leq \bar{q}^g_{n,\phi}, \forall n, \phi \\
(p^g_{n,\phi})^2 + (q^g_{n,\phi})^2 & \leq \bar{s}^g_{n,\phi}, \forall n, \phi \\
v & \leq \text{dg}(V_n) \leq \bar{v}, \forall n \\
\text{dg}(L_n) & \leq \bar{l}, \forall n
\end{align*}
\end{align*}
\]

- Relaxation is exact (constraint satisfied with equality) under practical conditions
Linear approximation for multiphase grids

• Ignore losses to get approximate power conservation

\[ \text{dg} (S_n - Z_n L_n) + s_n = \sum_{k: \text{n} \rightarrow \text{k}} \text{dg} (S_k) \quad \Rightarrow \quad \sigma_n + s_n = \sum_{k: \text{n} \rightarrow \text{k}} \sigma_k \]

• Voltage drop requires approximating the full matrix \( S_n \)

\[ V_n = V_{\pi n} + Z_n L_n Z_n^H - S_n Z_n^H - Z_n S_n^H \]

• Assuming approximately balanced voltages (and currents)

\[ v_{\pi n} \simeq V_{\pi n} \alpha, \quad i_n \simeq I_n \alpha \quad \alpha = \begin{bmatrix} 1 \\ \alpha^* \\ \alpha \end{bmatrix}, \quad \alpha = e^{j2\pi/3} \]

• Power flow matrix can be approximated as

\[ S_n = \alpha \alpha^H \text{dg} (\sigma_n) \]

\[ \text{dg}(V_n) = \text{dg}(V_{\pi n}) - \text{dg}(\alpha \alpha^H \text{dg}(\sigma_n) Z_n^H) - \text{dg}(Z_n \text{dg}(\sigma_n)^* \alpha \alpha^H) \]
Inter-phase coupling

- Simplify approximate voltage drop using the property
  \[dg\ (A\ dg(x)B) = (A \odot B^\top)x, \odot: \text{entry-wise (Hadamard) product}\]

- Approximate voltage drop
  \[v_{\pi_n} - v_n \simeq 2\Re\ \{\bar{Z}_n \sigma^*_n\}, \text{ where } \bar{Z}_n = Z_n \odot \alpha^* \alpha^\top\]

\[
Z_n = \begin{bmatrix}
0.530 + 1.112i & 0.127 + 0.404i & 0.126 + 0.423i \\
0.127 + 0.404i & 0.545 + 1.043i & 0.133 + 0.374i \\
0.126 + 0.423i & 0.133 + 0.374i & 0.542 + 1.056i
\end{bmatrix}
\]

\[
\bar{Z}_n = \begin{bmatrix}
0.530 + 1.112i & 0.286 - 0.312i & -0.430 - 0.103i \\
-0.413 - 0.092i & 0.545 + 1.0429i & 0.258 - 0.303i \\
0.304 - 0.321i & -0.391 - 0.072i & 0.542 + 1.056i
\end{bmatrix}
\]

- How do complex injections affect voltage drops?

\[
\text{sign} [\Re \{\bar{Z}_n\}] = \begin{bmatrix}
+ & + & - \\
- & + & + \\
+ & - & +
\end{bmatrix}
\]

\[
\text{sign} [\Im \{\bar{Z}_n\}] = \begin{bmatrix}
+ & - & - \\
- & + & - \\
- & - & +
\end{bmatrix}
\]