Power Flow Solvers for Direct Current Networks

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Abstract—With increasing smart grid direct current (DC) deployments in distribution feeders, microgrids, smart buildings, and high-voltage transmission, there is a need for better understanding the landscape of power flow (PF) solutions as well as for efficient PF solvers with performance guarantees. This work puts forth three approaches with complementary strengths towards coping with the PF task in DC power systems. We consider a possibly meshed network hosting ZIP loads and constant-voltage/power generators. The first approach relies on a monotone mapping. In the absence of constant-power generation, the related iterates converge to the high-voltage PF solution, if one exists. To handle distributed renewable generators typically operating in constant-power mode, an alternative Z-bus method is studied. For bounded constant-power generation and demand, the analysis establishes the existence and uniqueness of a PF solution within a predefined ball. Moreover, the Z-bus updates converge to this solution. Third, an energy function approach shows that under limited constant-power demand, all PF solutions are locally stable. The derived conditions can be checked without knowing the system state. The applicability of the conditions and the performance of the iterative schemes are numerically validated on a radial distribution feeder and two meshed transmission systems under varying loading conditions.

Index Terms—Fixed-point iterations, DC power flow, high-voltage solution, energy function minimization.

I. INTRODUCTION

With rampant developments on both generation and loads, the concept of a fully DC grid is getting closer to becoming a reality. Advances in photovoltaics, storage systems, and fuel cells, are inherently more compatible with the DC technology. Several types of residential loads (electronics, home appliances, and lighting) are DC in nature, and currently exhibit AC/DC conversion losses [1]. DC designs to reduce energy losses in commercial facilities serving a large number of nonlinear electronic loads have been studied [2], [3]. Case studies have demonstrated that DC designs feature reduced power losses and increased maximum power delivery capability [4]. For power transmission, high-voltage DC technologies are already being deployed, while plans for a super grid connecting large-scale renewable resources across Europe have favored the DC option [5].

Along with implementation changes, the development of DC (potentially coexisting with AC) systems bring about the need for new analytical tools. At the heart of power system studies lies the power flow (PF) task, in which the operator specifies the power injection or voltage at each bus, and solves the associated nonlinear equations to find the system state. There is a rich literature on the AC power flow problem. In transmission systems, the existence of a PF solution has been studied for example in [6], [7]; and its multiplicity in [8], [9]. In distribution systems, the same questions have been addressed in [10], [11]. For solvers coping with the AC PF task, see the recent comprehensive survey [12].

Justified by the limited interest in the past, the literature on the DC version of the PF task is rather limited. Reference [13] provides sufficient conditions under which a PF solution with large voltage values exists. However, the analysis is confined to DC networks hosting solely constant-power components and no solver is developed. Conventional solvers, such as the Newton-Raphson and Gauss-Seidel methods, provide no global convergence guarantees and rely heavily on initialization. Moreover, these methods do not provide any insight on the existence, uniqueness, stability, and high-voltage property of the found solution. Alternative solvers could be broadly classified into numerical methods for solving equations and optimization-based techniques, as detailed next.

Fixed-point iterations can handle the PF task leveraging certain properties of the involved mapping: The contracting voltage updates of [11] can conditionally find a PF solution in AC grids with constant-power buses. Another contraction mapping has been advocated for lossless AC networks in [14], [15]. To account for networks hosting constant-injection and constant-impedance loads too (ZIP loads), a contracting update known as the Z-bus method has been analyzed for single- and multi-phase distribution feeders [16], [17]. The Z-bus method has also been adopted to DC grids with ZIP loads [18], though the analysis fails to ensure that the updates remain within a compact voltage space. Relying on a monotone rather than a contraction mapping, the iterates devised in [19] are shown to converge to the unique high-voltage PF solution for AC networks; yet the conditions are confined to networks of constant line resistance-to-reactance ratios.

The PF task can be handled through an optimal power flow (OPF) solver: The system state can be found by minimizing an auxiliary cost (e.g., system losses) over the PF specifications posed as equality constraints. Reference [20] develops a second-order cone program relaxation of the OPF problem in DC networks with exactness guarantees, while demand response in DC grids is posed as a convex optimization in [1]. DC OPF methods could handle the PF task presuming all injections are constant-power. Another possibility is to treat the PF equations as the gradient of a differentiable function, known as the energy function, and hence, pose the PF task as a minimization problem. Historically used for stability analysis, the energy function minimization technique has been recently geared towards the PF problem in AC systems [21]. However, the conditions ensuring the energy function is convex depend on the sought system state. The energy function proposed in [19] is proved to be convex at all PF solutions in AC networks with constant resistance-to-reactance ratios.

This work puts forth and contrasts three methods for solving the PF task in DC power systems. Section II reviews a system model including ZIP loads and generators, all connected via a possibly meshed network. The first DC PF solver is a
fixed-point iteration on squared voltages (Section III). Under relatively light constant-power generation, the involved mapping is monotone and thus, the iterates converge provably to the unique high-voltage solution. To handle networks with larger constant-power generation, we secondly study a fixed-point iteration known also as the Z-bus method (Section IV). This method is shown to contract within a ball of voltages, within which a unique PF solution exists. Third, we adopt a minimization approach and express the PF solution as the stationary point of an energy function (Section V). Unless there is high constant-power demand, the function is convex at all PF solutions, thus establishing their local stability. All convergence guarantees rely solely on the network and its loading. Then, the system operator can readily identify which of the three methods is most suitable before solving the PF task. The methods are numerically tested under different loading conditions on a radial distribution feeder and two meshed transmission systems in Section VI.

Regarding notation, column vectors (matrices) are denoted by lowercase (uppercase) boldface letters; calligraphic symbols are reserved for sets. The $n$-th element of $x$ is denoted by $x_n$, and the $(n,m)$-th entry of $X$ by $X_{nm}$. The operator $\text{dg}(x)$ returns a diagonal matrix with $x$ on its main diagonal, and $||x||_q := \left(\sum_{n=1}^N |x_n|^q \right)^{1/q}$ is the $q$-th norm of $x$. Symbols 1 and $e_n$ denote the all-ones and $n$-th canonical vectors.

### II. DC Power System Modeling

A DC power system having $N+1$ buses can be represented by a graph $G = (N^+, \mathcal{L})$, whose nodes $N^+ := \{0, \ldots, N\}$ correspond to buses, and its edges $\mathcal{L}$ to lines. The set of buses $N^+$ can be partitioned into the set of constant-voltage buses $V$, and its complement denoted by set $\mathcal{P} := N^+ \setminus V$. The slack bus is indexed by $n = 0$ and it belongs to set $V$; the remaining buses comprise the set $N$.

Generation units can be modeled in two ways depending on their rating, on whether they are interfaced through a DC/DC converter, and converter control. Larger generation units are typically modeled by a constant-voltage source connected in series with a resistance [22], [1]; see Fig. 1(a). This resistance captures either an actual resistance, or the result of droop inverter control [23]. Either way, the generator is sited at a $V$ bus of degree one. Alternatively, a generator can be represented as a constant-power injection, as it is customary for units operating under maximum-power point tracking [1].

Each electric load can be modeled as of constant power; constant impedance (here conductance); constant current; or combinations thereof. Therefore, loads are located at $\mathcal{P}$ buses. A single bus may be serving multiple loads and/or generators; see Figure 1. Apparently, zero-injection nodes are considered degenerate $\mathcal{P}$ buses. Under a hybrid setup, possible connections with AC networks can be implemented as constant-power or constant-voltage buses.

Let $\{v_n, i_n, p_n\}$ denote respectively the voltage, current, and power injected from bus $n$ to the system. By definition, if $n \in V$, the voltage $v_n$ is fixed. Otherwise, the current injected from bus $n \in \mathcal{P}$ to the system can be decomposed as

$$i_n = -\frac{\gamma_n}{v_n} - \frac{p_n}{v_n} - g_n v_n$$

(1)

where $\gamma_n > 0$ is its constant-current component; $\gamma_n$ is the constant-power consumption; and $g_n > 0$ is the constant-conductance load on bus $n$. If bus $n$ hosts several loads and/or generators, the previous symbols denote the aggregate quantities. By convention, the power $p_n$ is positive for loads, and negative for generators.

From Kirchoff’s current law, the current $i_n$ is expressed as

$$i_n = \sum_{m \in N^+} g_{nm}(v_n - v_m)$$

(2)

where $g_{nm}$ is the conductance of the line connecting buses $n$ and $m$; and $g_{nm} = 0$ if the two buses are not directly connected, that is $(n,m) \notin \mathcal{L}$. For notational convenience, set also $g_{nn} = 0$ for all $n$. Let us also define

$$g_n := \sum_{m \in N^+} g_{nm}.$$  

(3)

Combining (1) and (2) gives

$$g_n v_n = \sum_{m \in N^+} g_{nm} v_m - i_n - \frac{p_n}{v_n} - g_n v_n.$$  

(4)

Multiplying both sides of (4) by $v_n$, splitting the summation in the right-hand side (RHS) over $m \in \mathcal{P} \setminus \{n\}$ and $m \in V$, and rearranging provides

$$c_n v_n^2 = \sum_{m \in \mathcal{P}} g_{nm} v_m v_n + k_n v_n - p_n$$

(5)

where constants $c_n$ and $k_n$ are defined for all $n \in \mathcal{P}$ as

$$c_n := g_n + g_n^2$$

(6a)

$$k_n := \sum_{m \in V} g_{nm} v_m - i_n^2.$$  

(6b)

The PF problem can be now formally stated as follows. Given the line admittances $\{g_{nm}\}$ for all $(n,m) \in \mathcal{L}$; the ZIP load/generator components $\{i_n, p_n, \gamma_n\}$ for all $n \in \mathcal{P}$; and the fixed voltages $\{v_n\}_{n \in V}$, find the remaining voltages $\{v_n\}_{n \in \mathcal{P}}$ satisfying (5). Note that if $p_n = 0$ for all $n \in \mathcal{P}$, the PF equations can be converted to linear upon dividing (5) by $v_n$. Otherwise, these equations are quadratic in $v_n$, do not admit a closed-form solution, and hence call for iterative solvers.

In this context, this work puts forth three DC PF solvers: a monotone mapping; a contraction mapping; and an energy function technique. Our analysis reveals that each method features convergence and other desirable properties under different generation and load setups. Due to the latter fact, the three methods have complementary applicability.

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**Fig. 1.** Bus types (from left to right): (a) Voltage-plus-resistance generator model converted to a constant-voltage bus; (b) Constant-power generator or load; (c) Constant-conductance load; and (d) Constant-current load.
III. MONOTONE MAPPING

Fixed-point iterations are an efficient way of finding solutions to non-linear equations. The equations in (5) can be rearranged into a fixed-point iteration whose equilibrium point corresponds to a PF solution:

$$v_n = \sum_{m \in \mathcal{P}} \frac{g_{nm}}{c_n} v_m + \frac{k_n}{c_n} v_n - \frac{p_n^o}{c_n}.$$  \hspace{1cm} (7)

Introduce the squared voltages $u_n := v_n^2$ to rewrite (7) as

$$u_n = \sum_{m \in \mathcal{P}} \frac{g_{nm}}{c_n} u_m \sqrt{u_n} + \frac{k_n}{c_n} \sqrt{u_n} - \frac{p_n^o}{c_n}.$$  \hspace{1cm} (8)

If the squared voltages $\{u_n\}_{n \in \mathcal{P}}$ are collected in the $P$-length vector $u$, the solution to (8) coincides with the equilibrium of the fixed-point equation

$$u = f(u)$$

where the $n$-th entry of the mapping $f : \mathbb{R}_+^P \to \mathbb{R}_+^P$ is

$$f_n(u) := \sum_{m \in \mathcal{P}} \frac{g_{nm}}{c_n} u_m \sqrt{u_n} + \frac{k_n}{c_n} \sqrt{u_n} - \frac{p_n^o}{c_n}.$$  \hspace{1cm} (9)

One may wonder whether the iterations $u^{t+1} = f(u^t)$ solve the non-linear equations in (8). To answer this, let us confine our interest within the set

$$\mathcal{U} := \{u : u_1 \leq u \leq \pi 1\}$$  \hspace{1cm} (10)

where the inequalities are understood entry-wise. Focusing our attention within $\mathcal{U}$ complies with grid standards that regulate voltages within a range. We next provide conditions under which $f(u)$ is monotone within $\mathcal{U}$: A mapping $f(u)$ is monotone if $f(u) \geq f(u')$ for all $u, u' \in \mathcal{U}$ with $u \geq u'$.

**Theorem 1.** The mapping $f(u)$ is monotone in $\mathcal{U}$ if

$$i_n^o \leq \frac{p_n^o}{\sqrt{2\pi - \frac{u}{\pi}}} g_n$$  \hspace{1cm} (11)

for all $n \in \mathcal{P}$ with $i_n^o \geq \sum_{m \in \mathcal{M}} g_{nm} v_m$.

Theorem 1 (proved in Appendix A) asserts that $f$ is monotone if all constant-current loads are relatively small compared to the network constants $g_n$'s. As validated in Section VI for several benchmark systems, condition (11) is met in general even under constant-current loads. This is true even when voltages are allowed to lie within the unrealistically wide range of $\pm 50\%$ per unit (pu). In this case, the coefficient $\frac{u}{\sqrt{2\pi - \frac{u}{\pi}}}$ in (11) becomes as low as 0.125 and for a more realistic range of $\pm 10\%$ per unit (pu), this ratio is 0.64. The value of $g_n$ is usually much larger than 1 (e.g., it equals 98, 30, and 14 for the IEEE 123-bus, 118-bus, Polish 2,736-bus systems, respectively). On the other hand, the value of $i_n^o$ is usually smaller than 1, since the power base is selected as the rating of the largest generator.

Leveraging the monotonicity of $f$, we will next study the equilibrium of the iterations

$$u^{t+1} := f(u^t).$$  \hspace{1cm} (12)

Before that, let us define the high-voltage solution of the PF equations and present a fundamental result to be used later.

**Definition 1.** If there exists a $u_{hv} \in \mathcal{U}$ for which $u_{hv} = f(u_{hv})$ and $u_{hv} \geq u$ for all $u \in \mathcal{U}$ with $u = f(u)$, this PF solution will be termed the high-voltage solution.

**Lemma 1.** [19, Th. 4] Consider the continuous and monotone mapping $f : [a, b] \to [a, b]$, and define the set

$$\mathcal{X} := \{x : x \in [a, b], x \leq f(x)\}.$$

The mapping $f(x)$ has a fixed point $x^*$ satisfying $x^* \geq x$ for all $x \in \mathcal{X}$. Furthermore, the iterations $x^{t+1} = f(x^t)$ converge to $x^*$ if initialized at $x$.

A high-voltage solution may not necessarily exist. If it does, it is unique by definition. Using Lemma 1 and the monotonicity of $f(u)$, we next study the existence of a high-voltage solution along with its recovery.

**Theorem 2.** Assume there exists a solution to $u_{hv}$ in $\mathcal{U}$. If (11) and for all $n \in \mathcal{P}$ it holds that

$$\pi g_n^o + \sqrt{\pi} i_n^o + p_n^o \geq 0$$  \hspace{1cm} (13)

the updates of (12) converge to $u_{hv}$ if initialized at $u := \pi 1$.

Theorem 2 (shown in Appendix A) adopts results on the monotone mapping devised in [19]. The analysis in [19] presumes: i) power lines of equal resistance-to-reactance ratios; ii) a radial AC network; and iii) is confined to constant-power injections. Here, these results are naturally applied to DC systems. They are further extended to meshed networks and non-constant-power injections under mild conditions. Condition (13) is trivially met if all buses host only loads since then $\{g_n^o, i_n^o, p_n^o\}$ are all non-negative, and/or generators are modeled as constant-voltage buses. Then according to Theorem 2, the DC PF equations feature a high-voltage solution that can be reached by iterating (12).

Condition (13) fails however if any bus $n \in \mathcal{P}$ hosts a constant-power generator and no loads, since then $p_n^o < 0$. This corresponds to the relevant case of relatively small distributed renewable generation, such as rooftop solar panels, operating under maximum power point tracking. To handle such cases, a different fixed-point iteration is considered next.

IV. CONTRACTION MAPPING

This section presents an alternative to the iterations in (12). The PF equations can be arranged into a different fixed-point iteration after dividing (7) by $v_n$ to get for all $n \in \mathcal{P}$

$$v_n = \sum_{m \in \mathcal{P}} \frac{g_{nm}}{c_n} v_m + \frac{k_n}{c_n} v_n - \frac{p_n^o}{c_n v_n}.$$  \hspace{1cm} (14)

Solving (14) could be pursued through the fixed-point iteration

$$v_n^{t+1} := h(v^t)$$  \hspace{1cm} (15)

where $v := [v_1 \cdots v_P]^T$ and the $n$-th entry of $h$ is

$$h_n(v) := \sum_{m \in \mathcal{P}} \frac{g_{nm}}{c_n} v_m + \frac{k_n}{c_n} v_n - \frac{p_n^o}{c_n v_n}, \forall n \in \mathcal{P}.$$

If $p_n^o \geq 0$ for all $n \in \mathcal{P}$, the mapping $h(v)$ is monotone, and so the convergence if (15) is guaranteed by an analysis similar to Theorem 2. Nevertheless, to study the convergence of (15)
under constant-power generation \((p_n^c \leq 0)\), this section takes a different route.

Define the \(P \times P\) matrix \(G\) with entries
\[
G_{nm} := \begin{cases} \sum_{m \in P} g_{nm}, & n = m \\ -g_{nm}, & n \neq m \end{cases}
\]
which is the reduced grid Laplacian matrix obtained upon ignoring the buses in \(V\) and using conductances as line weights. The buses in \(P\) are assumed to form a connected graph. If not, each connected component of the \(P\)-induced graph can be considered separately.

Being a Laplacian matrix, \(G\) is an M-matrix and hence \(G \geq 0\). Moreover, because \(G\) is a reduced Laplacian matrix, it holds \(G \succ 0\); see [24]. Based on \(G\), introduce matrix
\[
Z := \left( G + dg \left( \left( g_n^0 + \sum_{m \in V} g_{nm} \right) v \right) \right)^{-1}.
\]
After rearranging, the iterations in (15) can be expressed as
\[
v^{t+1} = h(v^t) = Z[k - D(v^t)p]
\]
where \(k := [k_1 \ldots k_P]^T\); \(p := [p_0^1 \ldots p_0^P]^T\); and \(D(v) := dg^{-1}(v)\). The update of (17) is also known as the Z-bus iterations, and have been used for solving the PF task with ZIP loads for single- and multi-phase AC networks [16], [17]; and DC networks [18]. To study the convergence of (17), recall the notion of a contraction mapping.

**Definition 2.** A mapping \(h(x) : \mathbb{R}^p \to \mathbb{R}^p\) is a contraction over the closed set \(C \subseteq \mathbb{R}^p\), if for all \(x, \bar{x} \in C:\)
\begin{enumerate}[(p1)]
\item \(h(x) \in C\) (self-mapping property); and
\item \(\|h(x) - h(\bar{x})\|_q \leq \alpha \|x - \bar{x}\|_q\) with \(0 \leq \alpha < 1\) for the \(\ell_q\) vector norm (contraction property).
\end{enumerate}

If a contraction mapping \(h\) has an equilibrium \(x = h(x)\) in \(C\), the equilibrium is unique and can be reached by the updates \(x^{t+1} := h(x^t)\); see [25]. The next result shown in Appendix B provides conditions under \(h\) is a contraction.

**Theorem 3.** Define \(d := Zk\) with \(d := \min_n |d_n|\), and the set \(C := \{v : \|v - d\|_q \leq R\}\) for some \(R > 0\) and \(q \geq 1\).

The iterations in (17) converge to the unique PF solution in \(C\) under the conditions
\[
\begin{align*}
R &\leq d \\ R(d - R) &\geq \|Z\|_q \|p\|_q \\ (d - R)^2 &\geq \|Z\|_q \|p\|_q.
\end{align*}
\]

Conditions (C1)–(C3) ensure that the updates in (17) remain positive and that \(h(v)\) is a contraction mapping within \(C\). Each one of (C1)–(C3) introduces a range for \(R\). We next study when their intersection is non-empty, and the physical intuition behind this; see Appendix B for a proof.

**Lemma 2.** The radius of the \(\ell_q\)-norm ball \(C\) for the contraction mapping of Theorem 3 is confined within
\[
R \in (R, R^*):= \left( \frac{d - \sqrt{d^2 - 4\beta}}{2}, d - \sqrt{\beta} \right)
\]
where \(\beta := \|Z\|_q \|p\|_q\).

Unlike its AC counterpart of [16], Lemma 2 consolidates (C1)–(C3) into a simple single condition in (20). Moreover, Lemma 2 ensures both the existence and uniqueness of a PF solution within \(C\). The condition in (20) holds in networks with light constant-power injections (small \(\|p\|_q\), and so small \(\beta\) and/or sufficient constant-voltage generation (large \(d\)). Different from the analysis of Section III, Lemma 2 covers both positive (loads) and negative (generators) entries of \(p\).

The existence and uniqueness claims of Theorem 3 hold for all \(R \in (R, R^*)\) as explained in [16]: A larger \(R\) means that the solution is unique within a larger ball \(C\). On the other hand, a smaller \(R\) implies that the unique solution is closer to \(d\). This is of interest when one wants to characterize the PF solutions over different scenarios without having to solve the PF task for each individual scenario. In the degenerate case of no constant-power injections, we get \(\beta = R = 0\) and so the ball center \(d = Zk\) becomes the unique PF solution. Recall it was exactly the presence of constant-power injections that rendered the PF equations non-linear. On the computational side, Theorem 3 asserts that if \(d^2 \geq 4\beta\) the voltage updates of (17) converge linearly to a unique PF solution within \(C\).

As a final note, the analysis of the Z-bus method for low-voltage DC grids in [18] fails to ensure self-mapping, thus yielding loose conditions. Albeit it presumes voltages to lie within a compact space to prove contraction, the mapping is not shown to remain within this compact space. To establish convergence of (17) via the Banach fixed-point theorem [25], all three conditions (C1)–(C3) should be guaranteed.

**V. Energy Function**

As an alternative to iterative methods, this section presents a PF solver relying on energy function minimization. The idea is to find an energy function so that its stationary points correspond to the solutions of the nonlinear equations at hand [21]. By doing so, the PF task is posed as a minimization problem. The convexity of the energy function over a domain ensures stability, meaning that for small disturbances, the electric system will return to this solution after a disturbance is cleared [26], [27]. Moreover, the strict convexity of the energy function implies uniqueness of its minimizer, and hence uniqueness of the PF solution [19], [21].

To explain the energy function method, let us transform the voltage variables as \(\rho_n := \log u_n\) for all \(n \in P\). The PF equations in (8) can be equivalently expressed as
\[
c_n e^\rho_n - \sum_{m \in P} g_{nm} e^{\rho_m} + k_n e^{\rho_n} + p_n^c = 0.
\]

Collecting \(\rho_n\)’s in vector \(\rho\), we define the energy function as
\[
E(\rho) := \sum_{n \in P} \left[ c_n e^\rho_n - 2k_n e^{\rho_n} + p_n^c \rho_n - \sum_{m \in P} g_{nm} e^{\rho_m + \rho_n} \right].
\]

Setting the partial derivative \(\frac{\partial E}{\partial \rho_n}\) to zero yields (21). Then, a PF solution can be found as the stationary point of \(E(\rho)\). If \(E(\rho)\) is convex, a PF solution can be found minimizing \(E(\rho)\).
To characterize the convexity of $E(\rho)$, let us find its Hessian matrix $H$ whose $(n,m)$-th entry is defined as $H_{nm} := \frac{\partial^2 E}{\partial \rho_n \partial \rho_m}$. Given that (21) corresponds to the partial derivative $\rho$-wise, we get that

$$H_{nm} = \begin{cases} e^{\frac{\rho}{2}} \left(c_n e^{\frac{\rho}{2}} - \frac{\rho}{2} - \sum_{k \in \mathcal{P}} g_{nk} e^{\frac{\rho}{2}} \right), & n = m \\ -\frac{g_{nm}}{2} e^{\frac{\rho}{2}} + \sum_{k \in \mathcal{P}} g_{nk} e^{\frac{\rho}{2}}, & n \neq m \end{cases}$$

To simplify the analysis, introduce matrix $\hat{H}(\rho) := 2dg \left( e^{-\frac{\rho}{2}} \right) H(\rho) \left( e^{-\frac{\rho}{2}} \right)$. Matrix $H(\rho)$ is positive definite if and only if $\hat{H}(\rho)$ is positive definite. Let $\lambda(A)$ denote the minimum eigenvalue of a symmetric matrix $A$. We next characterize the set of voltages for which $\hat{H}(\rho) \geq 0$, or equivalently $\lambda(\hat{H}(\rho)) \geq 0$.

**Theorem 4.** The energy function $E(\rho)$ is convex in $U$ if

$$[k_n]_+ \leq \sqrt{U} \left( \lambda(G) + 2c_n - \left( \frac{\pi}{U} + 1 \right) \sum_{m \in \mathcal{P}} g_{nm} \right)$$

for all $n \in \mathcal{P}$ where $[k_n]_+ := \max\{k_n, 0\}$.

Theorem 4 provides a sufficient condition for $E(\rho)$ to be convex in $U$. The proof of Theorem 4 is deferred to Appendix C. If the condition holds with strict inequality, the function is strictly convex and so there is a unique PF solution in $U$. Perhaps not surprisingly, this condition is hard to meet, but the convexity of $E(\rho)$ can be checked in a subset of $U$.

If a PF solution exists, one may be interested in studying the convexity of $E(\rho)$ around this solution. By continuity, $E(\rho)$ will be convex in a neighborhood, and so this solution is deemed as locally stable. The next lemma studies the convexity of $E(\rho)$ at a PF solution; see Appendix C for its proof.

**Lemma 3.** The energy function $E(\rho)$ is convex at any PF solution in $U$ if

$$[p_n^o]_+ \leq U \left( \lambda(G) + g_o + \sum_{m \in \mathcal{V}} g_{nm} \right)$$

for all $n \in \mathcal{P}$ where $[p_n^o]_+ := \max\{p_n^o, 0\}$.

The condition in (22) does not depend on the state $\rho$. The worst-case scenario occurs when bus $n$ is not connected to any constant-voltage bus and has high constant-power demand. In per unit, the quantity $[p_n^o]_+$ is much smaller than one, while $\lambda(G)$ is larger. Therefore, condition (22) holds for a wide range of practical cases as confirmed by the tests of Section VI.

**VI. NUMERICAL TESTS**

Our DC PF solvers were tested using the IEEE 123-bus radial distribution feeder; the IEEE 118-bus meshed transmission network; and the Polish 2,736-bus transmission system to test scalability. The multiphase 123-bus feeder was converted to its single-phase equivalent. To obtain a DC power network, line reactances were ignored in all three systems. For the 123-bus system, the nominal ZIP loading was maintained to its benchmark values, and only the substation was modeled as a constant-voltage bus. For the 118-bus and 2,736-bus systems, the nominal (constant-power) loading was separated into ZIP components by 30% constant-conductance, 30% constant-current, and 40% constant-power at nominal voltage of 1 pu. Generation units in the 118-bus and 2,736-bus systems were treated as constant-voltage buses at the nominal voltage.

We first tested the conditions (11)–(13) related to the monotone iterations of (12). Loads and generation were scaled by $0 – 200\%$ of their nominal values within a voltage regulation of up to $\pm 50\%$. All three systems satisfied (11). Condition (13) was satisfied too, since all generators were modeled as constant-voltage. To capture a scenario of distributed constant-power generation, we flipped the sign of all constant-power components. Condition (13) was still met for the 118- and the 2,736-bus systems, but not the 123-bus feeder.

We then tested condition $d^2 \geq 4\beta$ of Lemma 2. For the 123-bus system, condition (19) was met for $0 – 97\%$ of nominal load. The values for $\beta$, $d$, $G$, and $R$ are listed in Table I.

$$\begin{array}{|c|c|c|c|}
\hline
p^o & p^o/2 & p^o \\
\hline
\beta & 0.00 & 0.12 & 0.25 \\
\hline
d & 0.98 & 0.98 & 0.98 \\
\hline
G & 0.00 & 0.30 & - \\
\hline
R & 0.98 & 0.63 & - \\
\hline
\end{array}$$

The conditions ensuring convexity were also examined. Not surprisingly, the global convexity condition of Th. 4 did not hold for any system. However, condition (22) for local convexity was satisfied for all three systems within reasonable voltage ranges. The difference between the RHS and LHS of (22) is depicted in Fig. 2. The plots agree with the intuition that for increased loading and wider voltage ranges, the difference becomes smaller and the system may become less stable.

The monotone iterations of (12) and the Z-bus iterations of (15) were implemented under the benchmark conditions. The running times for reaching a relative error $\|v^{t+1} – v^t\|_2$ and $\|u^{t+1} – u^t\|_2$ of $10^{-6}$ are compared in Table II. The timing includes the matrix inversion for finding $Z$ needed in both cases. The two methods converged to the same state. By and large, the Z-bus updates were faster with the advantage becoming more significant with increasing system size.

Figure 3(a) shows the convergence of the Z-bus method for the 123-bus system at normal loading for three initializations. The method converged even though condition $d^2 \geq 4\beta$ of Lemma 2 was violated. That was confirmed even for loadings between 100 – 500% initialized at $d$ as shown in Fig. 3(b). The convergence rate is linear in log-scale (R-linear) with a slope decreasing with loading. Figure 3(c) shows the convergence of the monotone iterations of (12) again for loading $100 – 500\%$ initialized at $\pi = 1$ with $\pi = 2$. Notice that conditions (11)–(13) hold for all loading cases, so that the monotone method is guaranteed to converge to the high-voltage solution in this case. The convergence rate seems to be log-linear as well, though its slope is less than that of the Z-bus method.
Fig. 2. The difference between the RHS and LHS of (22): Positive values mean the energy function is convex (system stability) at all PF solutions within \( \mathcal{U} \).

![Figures showing voltage range and loading percentage](image)

**Fig. 3.** Left: Convergence of (15) for different initializations. Center and Right: Convergence of (15) and (9) respectively for 100 – 500% loading.

![Convergence plots](image)

<table>
<thead>
<tr>
<th>System</th>
<th>Monotone updates (12)</th>
<th>Z-bus updates (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEEE 118-bus</td>
<td>0.283</td>
<td>0.007</td>
</tr>
<tr>
<td>IEEE 123-bus</td>
<td>0.593</td>
<td>0.007</td>
</tr>
<tr>
<td>Polish 2,736-bus</td>
<td>1,111.53</td>
<td>1,223</td>
</tr>
</tbody>
</table>

**TABLE II**

**RUNNING TIME [SEC]**

**VII. CONCLUSIONS**

We have considered the DC PF task for possibly meshed networks hosting ZIP loads; (large) constant-voltage generators; and (smaller) constant-power generators. Under no constant-power generation, the suggested monotone mapping finds the high-voltage PF solution. Under limited constant-power generation and demand, the Z-bus iterations converge to a PF solution. The latter solution is known to exist and be unique within a predefined ball. For limited constant-power demand, the energy function minimization perspective has established that all PF solutions are locally stable. Interestingly, the first method operates on the space of squared voltages; the second on voltages; and the third on the logarithm of voltages. Numerical tests have demonstrated that the iterates converge to the same PF solution even when the conditions fail. Nevertheless, the analysis attributes different critical features to this solution. The convexity of the energy function and so the stability of the PF solution seems to be less sensitive for the radial feeder. For a few hundreds of buses, the monotone mapping and the Z-bus iterates seem to be comparable in terms of execution time. Yet for networks having thousands of buses, the latter has an indisputable advantage.

**APPENDIX A**

**Proof of Theorem 1:** Mapping \( f(u) \) is monotone in \( \mathcal{U} \) if and only if

\[
  f(u + \alpha e_n) \geq f(u), \quad \forall \ n \in \mathcal{P}
\]

for all \( \alpha \geq 0 \) such that \( u + \alpha e_n \in \mathcal{U} \), and \( e_n \) is the \( n \)-th column of the identity matrix of size \( P \).

Consider the condition in (23) for a particular \( n \in \mathcal{P} \). Since all but the \( n \)-th entries remain unchanged between \( u + \alpha e_n \) and \( u \), it is not hard to see that

\[
  f_m(u + \alpha e_n) - f_m(u) = g_{mn} \left( \sqrt{u_m(u_n + \alpha)} - \sqrt{u_m u_n} \right) \geq 0 \quad \forall m \neq n.
\]

Hence, the mapping \( f(u) \) is monotone in \( \mathcal{U} \) if and only if \( f_n(u + \alpha e_n) \geq f_n(u) \) for all \( n \in \mathcal{P} \). From the definition of \( f_n(u) \), it follows that

\[
  f_n(u + \alpha e_n) - f_n(u) = \sum_{m \in \mathcal{P}} g_{mn} \left( \sqrt{(u_n + \alpha) u_m} - \sqrt{u_n u_m} \right)
\]
\[ + \frac{k_n}{c_n} \left( \sqrt{u_n + \alpha} - \sqrt{u_n} \right). \] (24)

Since the square root is a concave function, the differences \(\sqrt{(u_n + \alpha)u_m} - \sqrt{u_nu_m}\) for \(m \in \mathcal{P} \setminus \{n\}\) appearing in the RHS of (24) can be lower bounded as
\[
\sqrt{(u_n + \alpha)u_m} - \sqrt{u_nu_m} \geq \frac{\alpha}{2} \sqrt{\frac{u_m}{u_n + \alpha}} \geq \frac{\alpha}{2} \sqrt{\frac{u_m}{2\pi - u}}
\]
since \(\pi - \frac{u}{\alpha} \geq \alpha\) to ensure \(\alpha + \alpha e_n \in \mathcal{U}\). Therefore, the first summand in the RHS of (24) is positive for all \(n \in \mathcal{P}\).

Focus next on the second term in the RHS of (24). If \(k_n < 0\) or equivalently \(i_n^o > \sum_{m \in \mathcal{P}} g_{nm} \sqrt{u_m} \), the concavity of the square root provides the lower bound
\[
\frac{k_n}{c_n} \left( \sqrt{u_n + \alpha} - \sqrt{u_n} \right) \geq \frac{\alpha k_n}{2c_n} \frac{1}{\sqrt{u_n}} \geq \frac{\alpha k_n}{2c_n} \frac{1}{\sqrt{\alpha}}.
\]
Plugging the two previous bounds into (24) and because \(g_{nm}\) and \(c_n\) are positive by definition, it follows that
\[
f_n(u + \alpha e_n) - f_n(u) \geq \frac{\alpha}{2c_n} \left( \sqrt{\frac{u}{2\pi - u}} \sum_{m \in \mathcal{P}} g_{nm} \sqrt{u_m} \right.
\]
\[
\left. + \frac{k_n}{c_n} \left( \sqrt{u_n + \alpha} - \sqrt{u_n} \right) \right) \geq \frac{\alpha}{2c_n} \left( \sqrt{\frac{u}{2\pi - u}} \sum_{m \in \mathcal{P}} g_{nm} \sqrt{u_m} + \frac{k_n}{c_n} \left( \sqrt{u_n + \alpha} - \sqrt{u_n} \right) \right).
\]
(25)

Since \(\alpha\) and \(c_n\) are positive, the monotonicity of \(f(u)\) is ensured if the quantity in the parentheses of (25) is non-negative. Plugging the definition of \(k_n\) from (6), the quantity in the parentheses becomes
\[
\sqrt{\frac{u}{2\pi - u}} \sum_{m \in \mathcal{P}} g_{nm} \sqrt{u_m} + \frac{\alpha k_n}{2c_n} \frac{1}{\sqrt{u_n}} \geq \frac{\alpha k_n}{2c_n} \frac{1}{\sqrt{\alpha}}.
\]
(26)

where the first inequality follows because \(u_m \geq \sqrt{\alpha}\), and the second inequality stems from \(\pi - \frac{u}{\alpha} \geq \alpha\) and the definition of \(g_n\) in (3). The condition in (11) guarantees that the RHS of (26) is non-negative for all \(n \in \mathcal{P}\) with negative \(k_n\).

If \(k_n \geq 0\), then \(\frac{k_n}{c_n} \left( \sqrt{u_n + \alpha} - \sqrt{u_n} \right) \geq 0\) holds trivially, and \(f_n(u + \alpha e_n) \geq f_n(u)\) from (24). For this reason, buses in \(\mathcal{P}\) with \(k_n \geq 0\) do not appear in the conditions of Th. 1.

**Proof of Theorem 2:** From Theorem 1, the condition in (11) guarantees \(f\) is monotone in \(\mathcal{U}\). Let \(u_s \in \mathcal{U}\) be a PF solution so \(f(u_s) = u_s\). We next show that \(f(u) \leq u\) under (13). By the definitions of \(c_n\) and \(f_n\) in (9):
\[
c_nu - f_n(u) = \frac{k_n}{c_n} \left( \sqrt{u_n + \alpha} - \sqrt{u_n} \right).
\]
(24)

for all \(n \in \mathcal{P}\). If the last quantity is non-negative for all \(n\), then \(f(u) \leq u\) follows.

The latter shows that \(f\) maps \([u_s, \bar{u}]\) to \([u_s, f(u)] \subset [u_s, \bar{u}]\).

Invoking Lemma 1 with \(\alpha = u_s\) and \(b = \bar{u}\) yields that the iterations in (12) initialized at \(u\) converge to a PF solution \(u_{hv}\) satisfying \(u_{hv} \geq u\) for all \(u \in [u_s, \bar{u}]\). Hence, the equilibrium \(u_{hv}\) is in fact the high-voltage power flow solution.

**Appendix B**

**Proof of Theorem 3:** For the subsequent analysis, a lower bound on voltages is needed. Since \(|\mathbf{v} - \mathbf{d}|_q^o \leq \mathbf{R}\) for all \(\mathbf{v} \in \mathcal{C}\), it follows that \(|\mathbf{v} - \mathbf{d}|_\infty \leq R\) for all \(\mathbf{v} \in \mathcal{P}\). Combining the latter with the reverse triangle inequality yields
\[
v_n \geq |d_n| - R, \quad \forall n \in \mathcal{P}.
\]
(27)

Under (C2), the RHS of (27) is positive, and thus, a non-trivial bound on voltages has been obtained.

For \(h(v)\) to satisfy the self-mapping property, we need to show that \(\|h(v) - d\|_q \leq R\) holds for all \(v \in \mathcal{C}\). Using the sub-multiplicative property of norms
\[
\|h(v) - d\|_q = \|ZD(v)p - ZD(v)p\|_q \leq \|Zv\|_q \cdot \|D(v) - D(v)\|_q \leq \|p\|_q.
\]
(28)

For a diagonal matrix \(\|d^q(x)\| = \max_q |x_n|\) for all \(q \geq 1\) (see e.g., [28, Th 5.6.37]).

Then from (27) we get
\[
\|D(v)\|_q = \left( \min_{n} |v_n| \right)^{-1} \leq (d - R)^{-1}.
\]

Plugging the latter into (28) renders condition (C2) sufficient for ensuring \(h(v) \in \mathcal{C}\).

Let us now upper bound the mapping distance:
\[
\|h(v) - h(\tilde{v})\|_q = \|ZD(v)p - ZD(\tilde{v})p\|_q \leq \|Z\|_q \cdot \|p\|_q \cdot \|D(v) - D(\tilde{v})\|_q \leq \|Z\|_q \cdot \|p\|_q \cdot \max_n \left\{ \frac{|v_n - \tilde{v}_n|}{v_n \tilde{v}_n} \right\}
\]
\[
\leq \|Z\|_q \cdot \|p\|_q \cdot \max_n |v_n - \tilde{v}_n| \leq \|Z\|_q \cdot \|p\|_q \cdot \frac{|v - \tilde{v}|_\infty}{|d - R|^2} \leq \|Z\|_q \cdot \|p\|_q \cdot \frac{|v - \tilde{v}|_\infty}{(d - R)^2} \leq \|Z\|_q \cdot \|p\|_q \cdot \frac{|v - \tilde{v}|_\infty}{(d - R)^2},
\]
where the third inequality comes from (27). Given the last bound, condition (C3) guarantees that the contraction property holds for \(\alpha = \|Z\|_q \cdot \|p\|_q/(d - R)^2\).

**Proof of Lemma 2:** From (C2), the radius \(R\) should satisfy \(R^2 - dR + \beta \leq 0\). To get a non-empty feasible range for \(R\), the previous convex quadratic should have a positive discriminant, i.e., \(d^2 \geq 4\beta\). Then \(R\) lies in the range between the roots of the quadratic as
\[
R \in \left[ \frac{d - \sqrt{d^2 - 4\beta}}{2}, \frac{d + \sqrt{d^2 - 4\beta}}{2} \right].
\]
(29)

Condition (C3) yields that \(|d - R| > \sqrt{\beta}\). Because of (C1), the latter simplifies as \(R < d - \sqrt{\beta}\), thus tightening (C1) as
\[
R \in \left[ 0, d - \sqrt{\beta} \right].
\]
(30)
The radius $R$ should satisfy both (29) and (30). For the lower side, it is not hard to see that because $\beta \geq 0$
\[
d - \sqrt{d^2 - 4\beta} \geq 0.
\]
For the upper side and since $d \geq 2\sqrt{\beta}$, one can write
\[
d^2 - 4\beta = \left(d - 2\sqrt{\beta}\right)\left(d + 2\sqrt{\beta}\right) \geq (d - 2\sqrt{\beta})^2.
\]
From (31), it follows that
\[
d + \sqrt{d^2 - 4\beta} \geq d + (d - 2\sqrt{\beta}) = d - \sqrt{\beta}.
\]
Combining the two sides yields the range of (19). Using (31) and because $d^2 \geq 4\beta$, we obtain that
\[
d - \sqrt{d^2 - 4\beta} \geq \frac{d - (d - 2\sqrt{\beta})}{2} \leq \sqrt{\beta} \leq d - \sqrt{\beta}
\]
so that the range of this lemma is not empty.

APPENDIX C

Proof of Theorem 4: Decompose $\tilde{H}(\rho)$ as $\tilde{H}(\rho) = \mathbf{G} + \mathbf{K}(\rho)$, where $\mathbf{K}(\rho)$ is a diagonal matrix with diagonal entries
\[
K_{nn}(\rho) := 2c_n - k_n e^{-\varphi_n} + \sum_{m \in \mathcal{P}} G_{nm} \left(e^{-\varphi_n} + 1\right).
\]
The minimum eigenvalue of $\tilde{H}$ satisfies [29, Th. 3.2.1]
\[
\lambda(\tilde{H}(\rho)) \geq \lambda(\mathbf{G}) + \lambda(\mathbf{K}(\rho)).
\]
Since $\mathbf{K}(\rho)$ is diagonal, its minimum eigenvalue is equal to its smallest diagonal entry. If the voltages lie in $\mathcal{U}$, a lower bound on $K_{nn}(\rho)$ can be obtained as
\[
K_{nn}(\rho) \geq 2c_n - \frac{|k_n|}{\sqrt{d}} + \left(\frac{\sum_{m \in \mathcal{P}} G_{nm}}{\sqrt{d}} + 1\right)\sum_{m \in \mathcal{P}} G_{nm}.
\]
Plugging (33) into (32) yields
\[
\lambda(\tilde{H}) \geq \lambda(\mathbf{G}) + 2c_n - \frac{|k_n|}{\sqrt{d}} + \left(\frac{\sum_{m \in \mathcal{P}} G_{nm}}{\sqrt{d}} + 1\right)\sum_{m \in \mathcal{P}} G_{nm}.
\]
For $\lambda(\tilde{H}) \geq 0$, the RHS of the last inequality must be positive, which is ensured by the condition of this theorem.

Proof of Lemma 3: If $\rho^o$ is a PF solution, it satisfies (21) for $n \in \mathcal{P}$. Exploiting this fact and from the definitions of $K_{nn}(\rho)$ and $c_n$, it follows that
\[
K_{nn}(\rho^o) = c_n - p_n^o e^{-\varphi_n^o} + \sum_{m \in \mathcal{P}} G_{nm}
\]
\[
= g_n^o - p_n^o e^{-\varphi_n^o} + \sum_{m \in \mathcal{V}} g_{nm}
\]
\[
\geq g_n^o - \frac{|p_n^o|}{u} + \sum_{m \in \mathcal{V}} g_{nm}.
\]
Using (32) again, condition (22) ensures $\lambda(\tilde{H}(\rho^o)) \geq 0$.

REFERENCES