On Multicast Interference Alignment in Multihop Systems

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***Abstract*—This paper investigates alignment schemes for multicast traffic over an equal path length multihop time-varying circularly symmetric fading channels. A finite field channel model is assumed, where the inputs and channel gains belong to the same field $\mathbb{F}_q$. The mechanism used in this paper combines elements of the alignment strategy developed by Nazer et al. [1] with the multihop unicast alignment strategy devised by Jeon et al. [2].**

***Index Terms*—Interference Channel, Alignment, Relay Networks, MAC channel**[1]

I. Introduction

The interference channel has received significant attention in recent years. In particular, the concept of *alignment* has been developed and used effectively to determine rate regions for different classes of interference channels. Alignment desires to minimize the dimension of the space spanned by the interference for a given signal space dimension [3]. There have been various (related) notions of interference alignment developed in literature [4], [1], [5]. Ergodic alignment is a relatively recent concept where time variations in the K-user interference channel are effectively used to reduce the space spanned by interference [1] . In [1], the authors investigate the K user interference channel with circularly symmetric fading, and find that a rate of $0.5 \log(1 + 2SNR)$ can be achieved for an additive Gaussian noise symmetric fading channel using ergodic alignment. Moreover, for a finite-field additive noise channel, the paper shows that ergodic alignment achieves capacity. Simultaneously, [2] analyzes unicast communication in a two-hop equal length network with time-varying channel states. Using store-and-forward relaying, the authors find that, for finite field symmetric fading channels, *point-to-point* channel capacity can be achieved when the number of relays exceeds the number of source-destination pairs. The general message-set problem for a single-hop network is investigated in [6], where the authors find that ergodic alignment can be generalized to the multicast case using discrete Fourier transform (DFT) matrices. This paper builds on the work in [2], [1], [6] to determine alignment strategies for a two-hop equal length multicast network.

[1]This work is supported by NSF grants CCF-0934924, CCF-0916713 and CCF-0905200.

A. Our Contributions

The following summarizes the two main contributions of this paper:

1) Determine alignment mechanisms for multi-hop communication for multicast networks when the number of relays exceeds the number of sources/destinations.
2) Combine elements of the multi-hop and ergodic alignment when the number of relays is smaller than the number of sources/destination.

The rest of this paper is organized as follows. In the next section, we describe the system model. In Section III, we describe the alignment scheme used for two-hop multicast networks. We conclude with Section IV.

II. System Model

A. Notation

In this paper, boldface is used to represent vectors and matrices. $rank(H)$ is used to denote the rank of a matrix H.

B. Network Description

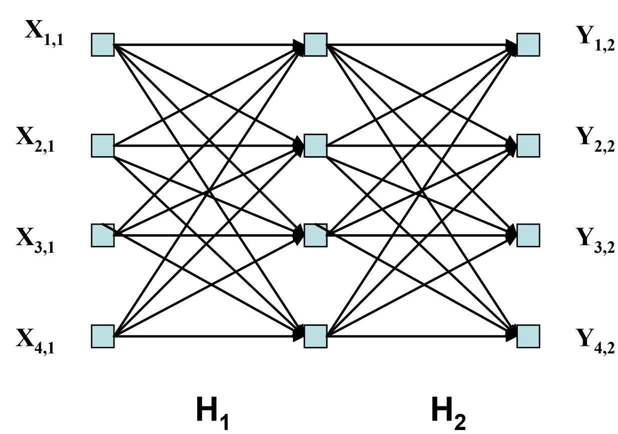

Fig. 1. A four node example of the network considered in this paper. Each receiver Y_{i2} desires one or more messages from the transmitters X_{i1}.

We consider a finite-field network consisting of 3 layers (labeled source, relay and destination layers respectively). Each layer is assumed to have K_i active nodes, that can communicate with nodes in the neighboring layers (see Figure III-A). For simplicity, we consider a system where the source and destination layers consist of $K_1 = K_3 = K$. The first hop is characterized by the linear function

$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{x}_1$$

where all variables are vectors over $\mathbb{F}_q$. Here, the channel input from the source $\mathbf{x}_1$ is a $K \times 1$ vector over $\mathbb{F}_q$ and $\mathbf{H}_1$ is an $K_2 \times K$ matrix.

The next hop is characterized by:

$$\mathbf{y}_2 = \mathbf{H}_2\mathbf{x}_2$$

where $\mathbf{x}_2$ is an $K_2 \times 1$ channel input from the relay and $\mathbf{H}_2$ is a $K \times K_2$ matrix. In general, $K_2 \neq K$, which leads to two different cases that must be handled separately:

1) An "overprovisioned" network with more relays than sources $K_2 > K$.
2) An "underprovisioned" network where $K_2 = \frac{K}{p}$, where $p \in \{2, 3, ...\}$.

The m-th hop transmission, $m \in \{1, 2\}$, is described as follows.

$$y_{j,m}[t] = \sum_{i=1}^{K} h_{j,i,m}[t]x_{i,m}[t]$$

Here the pair of indices (i, m) denote the ith transmitter, the pair (j, m) denote the j receiver, and $h_{j,i,m}[t] \in \mathbb{F}_q$ is the channel connecting Transmitter i to Receiver j at time t. Here, we assume that $h_{j,i,m}[t]$ are uniform and follow an i.i.d. fading model.

Note that our model does not incorporate an additive noise term at each hop of the network. This is for simplicity and to avoid distracting from the alignment scheme used in the paper. Additive noise terms can be incorporated (with suitable modifications to the achieved set of rates) along lines similar to [1].

C. Problem Statement

We consider a set of length n block codes. Let W_k be the message of the k-th source uniformly distributed over $\{1, 2, ..., 2^{nR_k}\}$, where R_k is the rate of the k-th source. A $(2^{nR_1}, ..., 2^{nR_K}; n)$ code consists of the following encoding, relaying, and decoding functions.

1) (Encoding) For $k \in \{1, ..., K\}$, the set of encoding functions of the k-th source is given by $\{f_{k,1,t}\}_{t=1}^{n}$: $\{1, 2, ..., 2^{nR_k}\} \rightarrow \mathbb{F}_q^n$ such that $x_{k,1}[t] = f_{k,1,t}(W_k)$ where $t \in \{1, ..., n\}$.
2) (Relaying) For $k \in \{1, ..., K_2\}$, the set of relaying functions of the (k, m)-th node is given by $\{f_{k,2,t}\}_{t=1}^{n}$: $\mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ such that $x_{k,2}[t] = f_{k,2,t}(y_{k,1}[1], ..., y_{k,1}[t-1])$ where $t \in \{1, ..., n\}$.
3) (Decoding) For $k \in \{1, ..., K\}$, the set of decoding functions of the k-th destination is given by g_k : $\mathbb{F}_q^n \rightarrow \{1, 2, ..., 2^{nR_k}\} \times \{1, 2, ..., 2^{nR_{k^\star}}\}$ such that $(\hat{W}_k, \hat{W}_{k^\star}) = g_k(y_{k,2}[1], ..., y_{k,2}[n])$ where $k^\star = (k + 1) \bmod K$.

The probability of error of the k-th destination is given by $P_{e,k}^{(n)} = \Pr((\hat{W}_k, \hat{W}_{k^\star}) \neq (W_k, W_{k^\star}))$. A set of rates $(R_1, ..., R_K)$ is said to be achievable if there exists a sequence of $(2^{nR_1}, ..., 2^{nR_K})$ codes with $P_{e,k}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$ for all $k \in \{1, ..., K\}$.

III. Multicast Alignment Strategies

The main insight behind our strategies is identical to that of [1], [2] - different channel states across stages and/or times can be combined to create an interference-free channel from source to destination. In the multicast network case, we desire to simultaneously sustain multiple access channel (MAC) between each destination and the corresponding sources from which it desires to receive messages.

A. Overprovisioned Network

We start with the case when the number of relays exceeds the number of sources and/or destinations in the network. In this case, the alignment scheme resembles the unicast case analyzed in [2] in that it aligns channels between stages using a store-and-forward strategy.

Definition 3.1: The complementary time instant t_k^c is the time instant in the second hop that satisfy

$$\mathbf{H}_2[t_k^c]\mathbf{H}_1[t_k] = \mathbf{L}$$

for a pre-specified matrix $\mathbf{L}$ and t_k is the k-th time instant in the first hop.

Here, the matrix $\mathbf{L}$ represents the connections desired between a destination and its desired sources. In essence, $\mathbf{L}$ has a non-zero entry in position (i, j) if the jth destination desires information from the ith source.

To simplify notation and present closed-form results, we consider the example symmetric multicast case first where

$$\mathbf{L} \triangleq \begin{bmatrix} 1 & l_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & l_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & l_{K-1} \\ l_K & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

Lemma 3.1: For every full rank matrix $\mathbf{H}_1[t]$ there exists a unique full rank matrix $\mathbf{H}_2[t^c]$ such that $\mathbf{H}_2[t^c]\mathbf{H}_1[t] = \mathbf{L}_K \ \forall t$. Also, we have that $(-1)^K \prod_{i=1}^{K} l_i \neq 1$.

Proof: We state the proof here for $K_2 = K$, but is easily straightforwardly extends to cases where $K_2 > K$. Recall from [7], [2] that for every full rank square matrix $\mathbf{H}_1$ there exists a unique inverse matrix $\mathbf{H}_1^{-1}$ which is full rank and square such that $\mathbf{H}_1^{-1}\mathbf{H}_1 = \mathbf{I}$ where $\mathbf{I}$ is the Identity matrix. For the given matrix $\mathbf{L}_K$, multiply both sides from left by $\mathbf{L}_K$ so we have $(\mathbf{L}_K\mathbf{H}_1^{-1})\mathbf{H}_1 = \mathbf{L}_K$ and let $\mathbf{H}_2 = \mathbf{L}_K\mathbf{H}_1^{-1}$. Recall [7] that for any matrix $\mathbf{A}$ and $\mathbf{B}$ we know that

$$rank(\mathbf{AB}) \leq min\Big(rank(\mathbf{A}), rank(\mathbf{B})\Big)$$

Under the given conditions that $(-1)^K \prod_{i=1}^{K} l_i \neq 1$ and $\mathbf{H}_1$ is full rank, we have $rank(\mathbf{H}_1) = rank(\mathbf{L}_K) = K$ and so from the previous inequality we get

$$K \leq min\Big(K, rank(\mathbf{H}_2)\Big)$$

Therefore $rank(\mathbf{H}_2) \geq K$ and since $\mathbf{H}_2$ is a $K \times K$ matrix. Then $\mathbf{H}_2$ must be full rank and unique. ■

If the pair $\mathbf{H}_1$ and $\mathbf{H}_2$ satisfies the condition $\mathbf{H}_2[t^c]\mathbf{H}_1[t] = \mathbf{L}_K \;\; \forall t$ then each destination $k \in \{1, 2, ..., K\}$ can receive messages from k-th and $k^\star$-th source in the absence of interference.

Note that, in general, the rank of the matrix $\mathbf{H}_1$ may not be K. However, addressing low rank matrices cannot be done in a manner similar to the unicast case studied in [2]. In the multiple unicast case, every receiver is associated with a unique transmitter, and thus, in the case of low rank channel matrices, we can still find transmitter-receiver pairs that can still simultaneously communicate with each other. In the multicast case, however, the source-destination sets are characterized by the matrix $\mathbf{L}$,and thus, when the matrix is low-rank, there is no guarantee that the rate between sources and destinations can be simultaneously sustained.

Note that low rank matrices may not be as difficult an issue to tackle for such networks. In a network where the number of relays exceeds the number of sources and/or destinations ($K_2 > K$), our interest is in determining a $K \times K$ submatrix of $\mathbf{H}_1$ that is full rank. As long as such a matrix exists, the conditions imposed by Lemma 3.1 are met and thus simultaneous communication between source and destination(s) is possible. In fact, the following lemma is this realization stated formally:

Corollary 3.2: For the case of an overprovisioned network (i.e. $K_2 > K$). The sum of the achievable rates of the K-user finite-field relay channel is given by

$$\sum_k R_k^s > \sum_k R_k^f \tag{2}$$

where R_k^f is the rate of k-th source in case when $K_2 = K$ and R_k^s is the rate of k-th source for the case when $K_2 > K$, where $k \in \{1, 2, ..., K\}$.

Proof: The statement follows directly from the fact that the addition of more than K nodes to the relay layer, increases the average rank of $\mathbf{H}_1$ and $\mathbf{H}_2$. Therefore, we have

$$\Pr(rank(\mathbf{H}_1) = i | K_2 > K) \geq \Pr(rank(\mathbf{H}_1) = i | K_2 = K) \forall i$$

therefore we have

$$\mathbf{E}(rank(\mathbf{H}_1) | K_2 > K) \geq \mathbf{E}(rank(\mathbf{H}_1) | K_2 = K)$$

■

Secondly, note that, if the alphabet size of the input is large enough, then the probability that a randomly generated $K \times K_2$ matrix not being full rank is diminishingly small (this forms the basis for randomized network coding arguments, for example, see [8]). Thus, the case of low rank matrices can be tackled by concatenating the input to enlarge the channel alphabet size. Finally, the low rank channel matrix case resembles the case of the underprovisioned network. This case is separately handled in the next section (Section III-B).

This leads to the main theorem for overprovisioned network:

Theorem 3.3: In an overprovisioned network with $K_2 \geq K$ and the multicast constraint matrix $\mathbf{L}$ as specified by Equation (1), the following set of rates are simultaneously achievable:

$$R_k + R_{k^*} \leq \log q \;\forall k \tag{3}$$

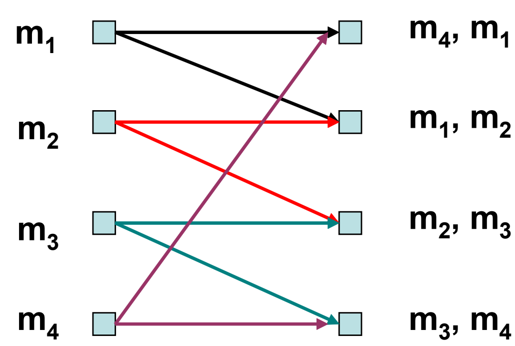

Fig. 2. A four node equivalent network after alignment. Four parallel MACs are created when the multicast requirement is given by (1).

Proof: This theorem is a straightforward consequence of Lemma 3.1. K simultaneously multiple access channels (MACs) can be sustained from the sources to the destinations. This, along with random coding arguments based on [9] gives us this result. ■

B. Underprovisioned Network

Lemma 3.4: For the underprovisioned network (i.e. $K_2 = \frac{K}{p}$) for every matrix $\mathbf{H}_1[t]$ with rank $r \leq \frac{K}{p}$ and $p \in \{2, 3, ...\}$, there exist a unique matrix $\mathbf{H}_2[t^c]$ with rank r such that

$$\mathbf{H}_2^A[t^c]\mathbf{H}_1^A[t] =$$

$$\begin{bmatrix} \mathbf{L}_r & B & \mathbf{0}_{r \times \frac{(p-1)K}{p}} \\ C & D & \mathbf{0}_{(\frac{K}{p} - r) \times \frac{(p-1)K}{p}} \\ \mathbf{0}_{\frac{(p-1)K}{p} \times r} & \mathbf{0}_{\frac{(p-1)K}{p} \times (\frac{K}{p} - r)} & \mathbf{0}_{\frac{(p-1)K}{p} \times \frac{(p-1)K}{p}} \end{bmatrix}$$

where

$$\mathbf{H}_1^A[t] = \begin{bmatrix} \mathbf{H}_1^\star[t] & \mathbf{0}_{K \times \frac{(p-1)K}{p}} \end{bmatrix},$$

$$\mathbf{H}_2^A[t^c] = \begin{bmatrix} \mathbf{H}_2^\star[t^c] \\ \mathbf{0}_{\frac{(p-1)K}{p} \times K} \end{bmatrix}$$

and $\mathbf{H}_1^A[t]$ and$\mathbf{H}_2^A[t^c]$ are the augmented matrices of $\mathbf{H}_1^\star[t]$ and $\mathbf{H}_2^\star[t^c]$, respectively, and $\mathbf{0}_{i \times j}$ is all zero matrix of size $i \times j$.

Proof: The addition of zeros columns to $\mathbf{H}_1^A$, result in $K \times K$ matrix with r independent columns. Similarly, the addition of zeros rows to $\mathbf{H}_2^A$, result in $K \times K$ matrix with r independent rows. Hence, the result follows directly from Lemma 3.1. ■

Theorem 3.5: For the case of underprovisioned network, where $K_2 = \frac{K}{p}$. The sum of the achievable rates of the K-user finite-field relay network are given by

$$\sum_k R_k^u = \frac{1}{p} \sum_k R_k^f. \tag{4}$$

Also, individual rates may also be characterized as

$$R_k^u = \frac{1}{p} R_k^f \ \forall k \tag{5}$$

where R_k^u is the rate of k-th source in case of underprovisioned network, and R_k^f represents any rates achieved on a network with $K_2 = K$ where $k \in \{1, 2, ..., K\}$..

Proof: This can be proved through two strategies. Strategy 1 is fairly specific in structure while Strategy 2 is general in that it may be applied more general classes of underprovisioned networks.

Strategy 1: In Strategy 1, we combine time instances (equivalently, perform symbol extension) to transform the system into an equivalent full rank channel transformation. This is very similar to alignment schemes Let $\mathbf{H}_1(1), \ldots, \mathbf{H}_1(p)$ denote p time instances of the channel H_1 which collectively are of rank K. Now, choose

$$[\mathbf{H}_2(1)\ \mathbf{H}_2(2)\ \ \ldots \mathbf{H}_2(p)]^T [\mathbf{H}_1(1)\ \mathbf{H}_1(2)\ \ \ldots \mathbf{H}_1(p)] = \mathbf{L}_K.$$

Note that such a collection of channel states $\mathbf{H}_2$ is unique as given by Lemma 3.1. Moreover, as each channel state $\mathbf{H}_2$ occurs equally often, such a collection of channel states is possible, and thus, over p time instances of the channel, K simultaneous multiple access channels can be supported between the sources and destinations. This gives us the result.

Strategy 2: Consider the first p time instants, we have a sequence of $\mathbf{H}_1[t_1], \mathbf{H}_1[t_2], ..., \mathbf{H}_1[t_p]$. Let the rank of $\mathbf{H}_1[t_i]$ is r_i where $i \in \{1, 2, ..., p\}$. Form the matrix $\mathbf{H}_1^A[t_1]$ and choose $\mathbf{H}_2^A[t_1^c]$ as in Lemma 3.4, so that we have $\mathbf{H}_2^A[t_1^c]\mathbf{H}_1^A[t_1]$ is a matrix of rank r_1 and with $K^2 - (\frac{K}{p})^2$ zeros. Using similar method as Lemma 3.4 construct $\mathbf{H}_1^A[t_2]$ and choose $\mathbf{H}_2^A[t_2^c]$ such that

$$\mathbf{H}_2^A[t_2^c]\mathbf{H}_1^A[t_2] = \begin{bmatrix} \mathbf{0}_{\frac{K}{p} \times \frac{K}{p}} & \mathbf{0}_{\frac{K}{p} \times r_2} & \mathbf{0}_{\frac{K}{p} \times (\frac{K}{p} - r_2)} & \mathbf{0} \\ \mathbf{0}_{r_2 \times \frac{K}{p}} & \mathbf{L}_{r_2} & B & \mathbf{0} \\ \mathbf{0}_{(\frac{K}{p} - r_2) \times \frac{K}{p}} & C & D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

which has rank r_2 and with $K^2 - (\frac{K}{p})^2$ zeros and $\mathbf{0}$ denote zeros matrices of suitable sizes that complete the size of $\mathbf{H}_2^A[t_2^c]\mathbf{H}_1^A[t_2]$ to $K \times K$. Repeat shifting by $\frac{K}{p}$ columns and rows in a similar fashion for constructing $\mathbf{H}_2^A[t_3^c]\mathbf{H}_1^A[t_3], \mathbf{H}_2^A[t_4^c]\mathbf{H}_1^A[t_4], ..., \mathbf{H}_2^A[t_p^c]\mathbf{H}_1^A[t_p]$ with corresponding ranks $r_3, r_4, ..., r_p$, respectively. Over the successive p time instants sum these constructed matrices to have $\mathbf{H}^S = \sum_{i=1}^p \mathbf{H}_2^A[t_i^c]\mathbf{H}_1^A[t_i]$ with rank $\sum_{i=1}^p r_i \leq K$. Recall $\mathbf{H}_2^\star[t^c]\mathbf{H}_1^\star[t]$ in Lemma 3.1 with rank $r = \sum_{i=1}^p r_i$. Conditioning on $rank(\mathbf{H}^S) = rank(\mathbf{H}_2^\star[t^c]\mathbf{H}_1^\star[t])$, we have $\Pr(\mathbf{H}^S) = \Pr(\mathbf{H}_2^\star[t^c]\mathbf{H}_1^\star[t])$. As $\mathbf{H}_2^\star[t^c]\mathbf{H}_1^\star[t]$ is constructed in one time instant and $\mathbf{H}^S$ is constructed in p time instant, this proves the result. ■

C. Generalized message sets

Our results can be generalized for case where each destination k is associated with source set S_k:

Lemma 3.6: For the overprovisioned relay case, for every $\mathbf{H}_1[t]$ of full rank there exists a full rank matrix $\mathbf{H}_2[t^c]$

$$\mathbf{H}_2[t^c]\mathbf{H}_1[t] = \mathbf{L}$$

where $\mathbf{L}$ is a matrix with non-zero entries characterized by the sets $S_k, \forall k$.

Proof: This lemma is a straightforward generalization of Lemma 3.1. ■

IV. Conclusions

In this paper, we determine alignment schemes for equal-length multihop multicast networks. Note that unequal length networks may have lower degrees of freedom as equal path networks, and thus our alignment strategies do not directly apply to them. For unequal path length networks, additional elements of ergodic alignment [1] must be included in the schemes proposed in this paper.

V. Acknowledgment

We thank Chris Slaughter for helpful discussions.

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