The Capacity Region of the Symmetric Linear Deterministic Interference Channel with Partial Feedback

Sy-Quoc Le\textsuperscript{1}, Ravi Tandon\textsuperscript{2}, Mehul Motani\textsuperscript{1}, and H. Vincent Poor\textsuperscript{3}

\textsuperscript{1}Department of ECE, National University of Singapore, Singapore.
\textsuperscript{2}Department of ECE, Virginia Tech, Blacksburg, VA, USA.
\textsuperscript{3}Department of EE, Princeton University, Princeton, NJ, USA.

Abstract—The linear deterministic interference channel (LD-IC) with partial feedback is considered. Partial feedback for the LD-IC models a scenario in which the top \( j \) most-significant-bits of the channel output of receiver \( j \) are received as feedback at transmitter \( j \), for \( j = 1, 2 \). The rationale for studying the LD-IC with partial feedback comes from the fact that it is a good approximation to the Gaussian interference channel with output feedback corrupted by additive white Gaussian noise (commonly referred to as noisy feedback). The main contribution of this paper is to characterize the capacity region of the symmetric LD-IC with partial feedback. The main ingredient of the proof is to obtain novel upper bounds on weighted rates \( 2R_1 + R_2 \) and \( R_1 + 2R_2 \).

I. INTRODUCTION

One of the most important issues for communication networks is that of interference management. Characterizing the capacity region of the two-user Gaussian interference channel (GIC) remains one of the fundamental unresolved problems in information theory. Recent breakthroughs in dealing with the capacity characterization of the GIC have made use of the linear deterministic interference channel (LD-IC) model \cite{1, 2}. The main idea behind these works is that an appropriately defined LD model can serve as a good approximation for the Gaussian channel. By gaining valuable insights from studying the LD-IC, the proof techniques and ideas can be lifted over to the GIC. The capacity region of the GIC has been characterized to within 1-bit in \cite{3}.

It is well known that feedback does not increase the capacity of the discrete memoryless point-to-point channel. However, feedback does increase the capacity of multi-user channels. The fact that feedback increases the capacity of the discrete memoryless multi-access channel (MAC) was shown by Gaarder and Wolf \cite{4}. Afterwards, Ozarow \cite{5} found the capacity region of the two-user Gaussian MAC with noiseless feedback. In one of recent work, Suh and Tse \cite{6} obtained an interesting result that noiseless feedback can provide significant capacity gains for the GIC. To understand the usefulness of feedback for the interference channel, consider the case of the very strong interference regime, in which the direct links are weaker than the cross (interference) links. In such a scenario, feedback can provide a substantial capacity increment by using the alternate path of \( T_X_1 \rightarrow R_X_2 \rightarrow T_X_2 \rightarrow R_X_1 \), i.e., the information intended from \( T_X_1 \) first reaches \( R_X_2 \), which is then received as feedback at \( T_X_2 \), which uses the strong cross (interference) link to reach the eventual destination at \( R_X_1 \). The approximate capacity region of the GIC with noiseless channel output feedback has been characterized in \cite{6} within 2-bits. The results in \cite{6} have been generalized to the case of the fully connected \( K \)-user IC \cite{7}, and the cyclic \( K \)-user IC \cite{8}. Vahid et al. considered an interesting generalization of \cite{6} by studying the two-user GIC with rate-limited feedback in \cite{9}. Rate-limited feedback refers to a setting in which the receiver can use all the information it has received so far and feed back information over an orthogonal channel of finite capacity (bit-pipe). Several interesting results for the GIC with rate-limited feedback are obtained in \cite{9}.

While rate-limited feedback may be useful in scenarios in which feedback links have good coding schemes to protect feedback signals from error, it places much complexity at the receiver’s side. As a result, this model is not appropriate when the complexity of the feedback design is a concern. In order to take some of these issues into account, this paper aims to investigate the model in which the feedback at transmitter \( j \) is a scaled and noisy (additive white Gaussian noise corrupted) version of the channel output received at receiver \( j \), for \( j = 1, 2 \). In particular, if the channel output at receiver \( j \) is \( Y_j \), then the feedback to transmitter \( j \) is \( Y_{F_j} = g_jY_j + Z_j \), for \( j = 1, 2 \) (see Figure 1). With the eventual goal of understanding the capacity region of the GIC with noisy feedback, we present a linear deterministic model with partial feedback. We show that the LD-IC with partial feedback serves as a good approximation for the GIC with noisy feedback.

The main result of this paper is to characterize the capacity region for the symmetric LD-IC with partial feedback. We note here that the sum-capacity of the LD-IC with partial feedback was characterized in our previous work \cite{10}. We illustrate through examples, that the sum-rate bounds derived in \cite{10} alone are not sufficient to characterize the capacity...
feedback consists of a sequence of encoding functions. Feedback code for the interference channel (IC) with noisy message sent back via the feedback link, which has the channel gain also applies to the feedback link. This means that the output must satisfy

In other words, for a code of block length \( T \),

\[
E = \sum_{t=1}^{T} |X_t|^2 \leq 1, j = 1, 2.
\]

This restriction also applies to the feedback link. This means that the output \( Y_{ji} \), for \( j = 1, 2 \), is normalized to unit power before it is sent back via the feedback link, which has the channel gain \( g_{Fj} \). Transmitter \( T_{xj} \), for \( j = 1, 2 \), wishes to communicate a message \( m_{j} \in \{1, 2, \ldots, M_j\} = W_j \) to receiver \( R_{xj} \). It is assumed that \( W_1 \) and \( W_2 \) are independent. An \( (M_1, M_2, T, P_e) \) feedback code for the interference channel (IC) with noisy feedback consists of a sequence of encoding functions

\[
f_j^* : W_j \times \{Y_{F1j}, Y_{F2j}, \ldots, Y_{F_,j-1}\} \rightarrow X_{ji}. \tag{7}
\]

Fig. 1. Gaussian IC with Noisy Feedback.

Fig. 2. Linear Deterministic IC with Partial Feedback.

\[
d_jT : \{Y_{j1}, Y_{j2}, \ldots, Y_{jT}\} \rightarrow \hat{W}_j \text{ for } j = 1, 2; \tag{8}
\]

such that \( \max \{P_e,1T, P_e,2T\} \leq P_e \), where \( P_e,1T \) and \( P_e,2T \) denote the average decoding error probabilities, which are computed as \( P_e,1T = E[P(\hat{w}_1 \neq w_1 | (w_1, w_2) \text{ were sent})] \). A rate pair \( (R_1, R_2) \) is achievable for the IC with noisy feedback if there exists an \( (M_1, M_2, T, P_e) \)-feedback code such that \( P_e \rightarrow 0 \) as \( T \rightarrow \infty \) and \( \frac{\log(M_i)}{T} \leq R_i \) and \( \frac{\log(M_2)}{T} \leq R_2 \). The capacity region of the IC with noisy feedback is defined as the closure of the set of all achievable rate pairs. With the goal of understanding the capacity region of the GIC with noisy feedback as defined above, we next describe the linear deterministic interference channel with partial feedback.

Using the deterministic model in [1], a non-negative integer \( n_{kj} \) is used to represent the channel gain from transmitter \( T_{xk} \) to receiver \( R_{xj} \) and it is given by \( n_{kj} = [\log h_{kj}]^+ \). Note that the vector of the Gaussian noise is captured by these representative numbers. Let \( q \) denote the maximum gain in the interference channel, i.e., \( q = \sup(n_{kj}) \). Thus, the transmitted signal from transmitter \( k \) at the time \( i \) will have a maximum of \( q \) bits visible to any receiver. Denote \( X_{ki} = [X_{ki1}, \ldots, X_{kiq}]^T \in F_q^q \), for \( k = 1, 2 \), where the leftmost bit is the most significant bit and the rightmost bit is the least significant bit. In this linear model, the effect of interference between various signals is captured as the superposition of those signals. At the time \( i \), the outputs at the receivers are given as

\[
Y_{1i} = S^{q-n_{11}} X_{1i} \oplus S^{q-n_{21}} X_{2i} \tag{9}
\]

\[
Y_{2i} = S^{q-n_{12}} X_{1i} \oplus S^{q-n_{22}} X_{2i}. \tag{10}
\]

where \( S \) is the a square shift matrix of size \( q \) and the operation is modulo 2 addition in \( F_2 \). Similarly, the channel gains for the feedback links can be represented by \( l_j \), for \( j = 1, 2 \), where \( l_j = [\log g_{Fj}]^+ \), and hence the feedback

\[
Y_{1i} = h_{11} X_{1i} + h_{21} X_{2i} + Z_{1i} \tag{11}
\]

\[
Y_{2i} = h_{12} X_{1i} + h_{22} X_{2i} + Z_{2i} \tag{12}
\]

\[
Y_{F1i} = g_{1} Y_{1i} + \tilde{Z}_{1i} \tag{13}
\]

\[
Y_{F2i} = g_{2} Y_{2i} + \tilde{Z}_{2i} \tag{14}
\]

where \( X_{ji} \) denotes the signal sent by transmitter \( j \), \( Y_{ji} \) denotes the output at receiver \( j \), \( Y_{Fji} \) denotes the feedback received at transmitter \( j \), for \( j = 1, 2 \), at time \( i \), for \( i \in \{1, 2, \ldots, T\} \), and \( \{Z_{ji}\}_{i=1}^T \) and \( \{\tilde{Z}_{ji}\}_{i=1}^T \) are independent, additive white Gaussian noise processes with zero means and unit variances. The forward channel gains \( \{h_{11}, h_{21}, h_{12}, h_{22}\} \) and the feedback channel gains \( \{g_{1}, g_{2}\} \) are assumed to be constant and known at all terminals. Average unit power constraints are imposed at each transmitter. In other words, for a code of block length \( T \), input sequences must satisfy \( \frac{1}{T} \mathbb{E}(\sum_{t=1}^{T} |X_t|^2) \leq 1, j = 1, 2 \). This restriction also applies to the feedback link. This means that the output \( Y_{ji} \), for \( j = 1, 2 \), is normalized to unit power before it is sent back via the feedback link, which has the channel gain \( g_{Fj} \). Transmitter \( T_{xj} \), for \( j = 1, 2 \), wishes to communicate a message \( m_{j} \in \{1, 2, \ldots, M_j\} = W_j \) to receiver \( R_{xj} \). It is assumed that \( W_1 \) and \( W_2 \) are independent. An \( (M_1, M_2, T, P_e) \) feedback code for the interference channel (IC) with noisy feedback consists of a sequence of encoding functions

\[
f_j^* : W_j \times \{Y_{Fj1}, Y_{Fj2}, \ldots, Y_{Fj, j-1}\} \rightarrow X_{ji}. \tag{7}
\]
signals at the transmitters are given as
\[ Y_{1i} = S^{q-l_1}Y_{1i}, \quad Y_{2i} = S^{q-l_2}Y_{2i}. \]  
(11)

Effectively, via the feedback links, transmitter \( j \) sees only the top \( l_j \) bits of the received signals \( Y_{ji} \) (see Figure 2). The focus of this paper is on the symmetric LD-IC in which \( m = n_{12} = n_{21}, n = n_{11} = n_{22}, \) and \( l = l_1 = l_2 \).

Given a triple \((n, m, l)\), we denote the capacity region with partial feedback by \( C^{\text{P-FB}}(n, m, l) \), which is the set of all achievable rate pairs \((R_1, R_2)\) with partial feedback. We find it useful to define forward, and feedback interference parameters respectively as follows
\[ \alpha = \frac{m}{n}, \quad \beta = \frac{l}{n}. \]
(12)
The forward interference parameter \( \alpha \) measures the normalized interference, whereas the feedback interference parameter \( \beta \) measures the level of normalized feedback.

### III. MAIN RESULTS

The main result of this paper is the capacity region for the symmetric interference channel with partial feedback.

**Theorem 1:**
\[ C^{\text{P-FB}}(n, m, l) \subseteq C^{\text{P-FB}}(n, m, l), \]
(13)
where the region \( C^{\text{P-FB}}(n, m, l) \) is the set of non-negative rate pairs \((R_1, R_2)\) that satisfy the following bounds
\begin{align*}
R_1 &\leq \max(n, m), \\
R_2 &\leq \max(n, m), \\
R_1 + R_2 &\leq (n - m)^+ + \max(n, m), \\
R_1 + R_2 &\leq 2 \max[(n - m)^+, m] + 2 \min[(n - m)^+, (l - \max(m, (n - m)^+))^+, m], \\
R_1 &\leq n + (l - n)^+, \\
R_2 &\leq n + (l - n)^+, \\
2R_1 + R_2 &\leq (n - m)^+ + \max(m, (n - m)^+) + \min[(n - m)^+, (l - \max(m, (n - m)^+))^+, m], \\
R_1 + 2R_2 &\leq (n - m)^+ + \max(m, (n - m)^+) + \min[(n - m)^+, (l - \max(m, (n - m)^+))^+, m].
\end{align*}

For each interference regime, we present the reverse inclusion as well, and thus we have the complete capacity characterization, i.e.,
\[ C^{\text{P-FB}}(n, m, l) = C^{\text{P-FB}}(n, m, l). \]
(14)

We illustrate these results through an example in which \( n = 6, m = 2 \) and \( l = 5 \), Figure 3 shows the capacity regions with no feedback, with full feedback, with rate-limited feedback of \( l = 5 \) bits, and with partial feedback of \( l = 5 \) bits. Several interesting observations are worth making: a) the capacity region with full feedback coincides with that of rate-limited feedback of \( l = 5 \) bits; b) the sum capacity is 10 bits/channel-use for full, rate-limited and partial feedback settings; and c) most importantly, the capacity region with partial feedback is strictly contained in the capacity region with full feedback and rate-limited feedback. It is here that we can clearly see the necessity of \( 2R_1 + R_2 \) and \( R_1 + 2R_2 \) bounds in characterizing the exact capacity region.

### IV. CONVERSE PROOFS

We first note the single user cut-set bounds given as
\[ R_1 \leq h(Y_1) = \max(n, m), \]
\[ R_2 \leq h(Y_2) = \max(m, n). \]

The bound \( R_1 + R_2 \leq (n - m)^+ + \max(n, m) \) is the same as the bound on the sum rate with full feedback as shown in [6]. The next bound on the sum-rate \( R_1 + R_2 \) and the fifth bound on \( R_1 \) and the sixth bound on \( R_2 \) were presented previously in [10].

In the remainder of this section, we therefore focus on the proof for the upper bound on \( 2R_1 + R_2 \). The proof for the bound on \( R_1 + 2R_2 \) follows in a similar manner.

Let \( S_{D_1} \) represent the top \( m \) bits of the first transmitter. When \( m \leq n \), it will be the top \( m \) bits out of \( n \) bits. When \( n < m \), it will represent all the bits from the first transmitter. Intuitively, \( S_{D_1} \) represents the \( m \) common information bits from the first transmitter, and is received by both receivers. Similarly, let \( S_{D_2} \) represent the top \( m \) bits for the second transmitter.

Furthermore, define \( X_{top} \), as the top \( \min(m, (2m - n)^+) \) bits of transmitter \( j \). In other words, \( X_{top} \) comprises the top \( (2m - n)^+ \) bits of the transmitter \( j \) when \( \frac{n}{2} \leq m \leq n \), and the top \( m \) bits when \( m > n \). No equivalent variable is defined in the case when \( m < \frac{n}{2} \). These two random variables mainly serve to explain the bounds in the weak interference regime and the moderately strong interference regime.

It is worthwhile to give examples of these four random variables for ease of reading. Consider the case of the very weak interference regime where \( 0 \leq m < \frac{n}{2} \). Consider Figure 5. In this case, \( S_{D_1} \) represents the region \((A_1, A_2), \)

![Fig. 3. Capacity regions under various feedback assumptions.](image-url)
and $S_{D_2}$ represents the region $(B_1, B_2)$. $X_{top_1}$ and $X_{top_2}$ are null regions in this regime.

Consider a second example. Consider the case of the weak interference regime where $\frac{m}{2} \leq m < \frac{2m}{3}$. Consider Figure 7. In this case, $S_{D_1}$ represents the region $(A_1, A_2, A_3)$, and $S_{D_2}$ represents the region $(B_1, B_2, B_3)$. $X_{top_1}$ is $A_1$ and $X_{top_2}$ is $B_2$ respectively.

We have

$$T(2R_1 + R_2 - p_e^T) \leq 2I(W_1; Y_1^T) + I(W_2; Y_2^T)$$

$$\leq I(W_1; Y_1^T Y_2^T) + I(W_1; Y_1^T Y_2^T | W_2) + I(W_2; Y_2^T)$$

$$= h(Y_1^T) + h(Y_2^T | Y_1^T) - h(Y_2^T | W_1) - h(Y_1^T | Y_2^T W_1)$$

$$+ h(Y_2^T | W_2) + h(Y_1^T Y_2^T | W_2) - h(Y_1^T Y_2^T | W_1 W_2)$$

$$+ h(Y_2^T | Y_2^T W_2) - h(Y_1^T Y_2^T | W_2) - h(Y_2^T | Y_2^T W_2)$$

$$= (a) h(Y_1^T) - h(Y_1^T | W_1) - h(Y_1^T | Y_2^T W_1) + h(Y_2^T | Y_2^T W_2)$$

$$+ h(Y_2^T | Y_2^T W_2) - h(Y_2^T | Y_2^T W_2)$$

$$= (b) h(Y_1^T) - h(Y_1^T | W_1) - h(S_{D_1} T X_{top_1} | Y_1^T W_2) + h(Y_2^T | Y_2^T W_2)$$

$$+ h(Y_2^T | Y_2^T W_2) - h(Y_2^T | Y_2^T W_2)$$

$$= (c) h(Y_1^T) - h(Y_1^T | W_1) - h(S_{D_1} T X_{top_1} | Y_1^T W_2)$$

$$+ h(Y_2^T | Y_2^T W_2) - h(Y_2^T | Y_2^T W_2)$$

$$\leq (d) h(Y_1^T) - h(Y_1^T | W_1) - h(S_{D_1} T X_{top_1} | Y_1^T W_2)$$

$$+ h(Y_2^T | Y_2^T W_2) - h(Y_2^T | Y_2^T W_2)$$

$$= (e) h(Y_1^T) - h(Y_1^T | W_1) - h(S_{D_1} T X_{top_1} | Y_1^T W_2)$$

$$+ h(Y_2^T | Y_2^T W_2) - h(Y_2^T | Y_2^T W_2)$$

$$= (f) h(Y_1^T) - h(Y_1^T | W_1) - h(S_{D_1} T X_{top_1} | Y_1^T W_2)$$

$$+ h(Y_2^T | Y_2^T W_2) - h(Y_2^T | Y_2^T W_2)$$

$$\leq (g) h(Y_1^T) - h(Y_1^T | W_1) - I(W_2; Y_1^T W_1)$$

$$+ I(S_{D_1} T X_{top_1} Y_2^T W_2 | Y_1^T W_1)$$

$$+ h(Y_2^T | S_{D_1} T X_{top_1} W_2) + h(Y_2^T | S_{D_1} T X_{top_1}) - h(S_{D_1} T Y_2^T W_2)$$

$$= h(Y_1^T) - h(Y_1^T | W_1) + I(S_{D_1} T X_{top_1} Y_2^T W_2 | Y_1^T W_1)$$

$$+ h(Y_2^T | S_{D_1} T X_{top_1} Y_2^T W_2) + h(Y_2^T | S_{D_1} T X_{top_1}) - h(S_{D_1} T Y_2^T W_2)$$

$$\leq (h) h(Y_1^T) - h(Y_1^T | W_1) + [h(Y_1^T | W_1)]$$

$$+ h(Y_2^T | S_{D_1} T X_{top_1} Y_2^T W_2) + h(Y_2^T | S_{D_1} T X_{top_1})$$

$$= h(Y_1^T) - h(Y_1^T | W_1) + [(h(Y_1^T | W_1)]$$

$$+ h(Y_2^T | S_{D_1} T X_{top_1} Y_2^T W_2) + h(Y_2^T | S_{D_1} T X_{top_1})$$

$$\leq \sum_{i=1}^{j} [h(Y_1^T) + h(Y_2^T | S_{D_2} Y_2^T | W_2) X_{top_1} W_2 X_{2,i}]$$

$$+ h(S_{D_2} | Y_2^T W_2 X_{2,i}) + h(X_{top_1} Y_2^T | S_{D_2} W_2)$$

$$+ h(Y_1^T | S_{D_1} S_{D_2}) + h(Y_2^T | S_{D_1} X_{top_1})$$

$$\leq \sum_{i=1}^{j} [h(Y_1^T) + h(Y_2^T | S_{D_2} Y_2^T | W_2) X_{top_1} W_2 X_{2,i}]$$

$$+ h(S_{D_2} | Y_2^T W_2 X_{2,i}) + h(X_{top_1} Y_2^T | S_{D_2} W_2)$$

$$+ h(Y_1^T | S_{D_1} S_{D_2}) + h(Y_2^T | S_{D_1} X_{top_1})$$

where (a) comes from the facts that $h(Y_1^T Y_2^T | W_2 W_1) = 0$, $h(Y_1^T | Y_2^T) = 0$, and $h(Y_2^T | Y_2^T) = 0$; (b) comes from the fact that $X_{j,i}$ is a function of $(Y_{j,i}^{-1} W_2)$, for $j = 1, 2$; (c) follows from the fact that $X_{top_1}^T$ is a function of $X_1^T$, which is in turn a function of $(W_1 Y_2^{-1} W_1)$. This is the crucial step; (d) and (e) come from the fact that more side information increases the entropy; (f) comes from the fact that more side information increases the mutual information; (g) comes from the fact that $S_{D_1}$ is a function of $(Y_2^T W_2)$; (h) follows from the fact that, given $(W_1, W_2)$, the entropy of any random variable is 0; (i) comes from the fact that $X_2,i$ is a function of $(Y_2^{-1} W_2)$; and (j) follows from the fact that $S_{D_2,i}$ is a function of $X_{2,i}$.

Case 1: $0 \leq m \leq \frac{2}{3}$

We have

$$h(Y_1^T) \leq n;$$

$$h(Y_{F_2}, i | X_{2,i} X_{top_1,i}) = (l - (l - m))^+;$$

$$h(X_{top_1,i} | S_{D_2,i}) = 0;$$

$$h(Y_{1,i} | S_{D_1,i} S_{D_2,i}) = n - m;$$

$$h(Y_{2,i} | S_{D_2,i} X_{top_1,i}) = h(Y_{2,i} | S_{D_2,i}) = n - m.$$ 

Thus, we have $2R_1 + R_2 \leq 3n - 2m + [l - (l - m)]^+.

Case 2: $\frac{2}{3} \leq m \leq n$

We have

$$h(Y_1^T) \leq n;$$

$$h(Y_{F_2}, i | X_{2,i} X_{top_1,i}) = (l - m)^+;$$

$$h(X_{top_1,i} | S_{D_2,i}) = (2m - n)^+;$$

$$h(Y_{1,i} | S_{D_1,i} S_{D_2,i}) = n - m;$$

$$h(Y_{2,i} | S_{D_2,i} X_{top_1,i}) = n - m.$$ 

Thus, we have $2R_1 + R_2 \leq 2n + [l - m]^+.

Case 3: $n \leq m$

We have

$$h(Y_1^T) \leq m;$$

$$h(Y_{F_2}, i | X_{2,i} X_{top_1,i}) = 0;$$

$$h(X_{top_1,i} | S_{D_2,i}) = m;$$

$$h(Y_{1,i} | S_{D_1,i} S_{D_2,i}) = 0;$$

$$h(Y_{2,i} | S_{D_2,i} X_{top_1,i}) = h(Y_{2,i} | S_{D_2,i}) = 0.$$ 

Thus, we have $2R_1 + R_2 \leq 2m$.

Combining the three cases, we have proven the bound on $2R_1 + R_2$.

V. Achievable Schemes with Partial Feedback

From the outer bounds in the previous section, we can determine the corner points for each of five regimes. If we can show those corner points are achievable, the capacity region is established for that particular regime. This is the approach we take in this section.
The rest of the operations are similar to that in the first time
min
and
A
min
Fig. 6. Encoding for corner point
m
n
The capacity region in this case is shown in Figure 4. It
E
B
223
2
2
Fig. 5. Encoding for corner point
(n, n−2m + \min(m, (l−(n−m))^+))
A
min

\begin{align*}
A_1 &: (n - \min\left(\frac{m}{2}, (l - (n-m))^+\right), \\
n - 2m + 3\min\left(\frac{m}{2}, (l - (n-m))^+\right)) \\
A_2 &: (n, n - 2m + \min(m, (l - (n-m))^+))
\end{align*}

Fig. 4. Capacity region for LD-IC for \( \alpha \in [0, \frac{1}{2}] \).

A. Very-Weak Interference: \( \alpha \in [0, \frac{1}{2}] \)

The capacity region in this case is shown in Figure 4. It is trivial to show that the points \((0, 0), (n, 0)\) and \((0, n)\) are achievable. Due to symmetry, we just need to show that the two points \((n, n - 2m + \min(m, (l - (n-m))^+))\) and \((n - \min\left(\frac{m}{2}, (l - (n-m))^+\right), n - 2m + 3\min\left(\frac{m}{2}, (l - (n-m))^+\right))\) are achievable.

Firstly, we are going to show how to achieve the point
\((n, n - 2m + \min(m, (l - (n-m))^+))\). The encoding scheme is shown in Figure 5. The set of \(n\) bits at transmitter \(T_{X_1}\), for each channel use, is divided into 5 encoding regions, \(A_1, A_2, ..., A_5\), of respective sizes \(\frac{m}{2}, \frac{m}{2}, n - 2m, \frac{m}{2}, \frac{m}{2}\). A similar partition is done at \(T_{X_2}\).

For every time slot, transmitter 1 always transmits \(n\) fresh information bits. In the first time slot, transmitter 2 transmits \(n - 2m\) + \(E_{21}\) fresh information bits in the regions \(B_3\) and \((B_4, B_5)\) as shown in the diagram. The size of \(E_{21}\) is \(\min(m, (l - (n-m))^+)\). Receiver 2 feeds back the top \(l\) bits, which include \(E_{21} \oplus E_{11}\). Thus, at the end of the first time slot, transmitter 2 can decode \(E_{11}\). In the second time slot, transmitter 2 relays \(E_{11}\) bits in the region \((B_4, B_5)\).

The rest of the operations are similar to that in the first time slot. Notice that \(E_{11}\) this time does not cause interference to receiver 1 as receiver 1 has already received those bits in the first time slot. At the same time, \(E_{11}\) is received cleanly at receiver 2. As a result, receiver 2 can decode the information \(E_{21}\) transmitted in the first time slot. By repeating those operations for all time slots, the corner point \((n, n - 2m + \min(m, (l - (n-m))^+))\) is achievable.

Next, we are going to show how to achieve the corner point \((n - \min\left(\frac{m}{2}, (l - (n-m))^+\right), n - 2m + 3\min\left(\frac{m}{2}, (l - (n-m))^+\right))\). The encoding scheme is shown in Figure 6. In the first time slot, transmitter 1 encodes \(n - \min\left(\frac{m}{2}, (l - (n-m))^+\right)\) fresh information bits which cover all of the regions \((A_1, A_3, A_4, A_5)\) and partially cover the region \(A_2\). The bottom of the region \(A_2\), of the size \(\min\left(\frac{m}{2}, (l - (n-m))^+\right)\) is left empty. Transmitter 2 transmits \(n - 2m + 3\min\left(\frac{m}{2}, (l - (n-m))^+\right)\) bits in the regions \(B_3, E_{211}, E_{212}, 223\). Notice that \(|E_{111}| = |E_{213}| = \min\left(\frac{m}{2}, (l - (n-m))^+\right)\), for \(j = 1, 2, 3\). At the end of the first time slot, via the feedback link, transmitter 1 can receive \(E_{211}\), and transmitter 2 can receive \(E_{111}\). In the second time slot, besides repeating the operation in the previous time slot, transmitter 1 relays the information in \(E_{211}\). In this time slot, receiver 1 can receive \(E_{211}\) cleanly, thus receiver 1 can resolve the interference in the previous time slot and decode those corresponding corrupted bits in the region \(A_1\) in the previous time slot. Notice that \(E_{211}\) does not cause interference to receiver 2 as it was received perfectly by receiver 2 in the previous time slot already. Besides repeating operation in the previous time slot, transmitter 2 relays the information in \(E_{111}\). Similarly, \(E_{111}\) is received cleanly by receiver 2, and thus helps receiver 2 resolve interference in the previous time slot. By repeating those operations, we can achieve the corner point \((n - \min\left(\frac{m}{2}, (l - (n-m))^+\right), n - 2m + 3\min\left(\frac{m}{2}, (l - (n-m))^+\right))\).

B. Weak Interference: \( \alpha \in \left[\frac{1}{2}, \frac{2}{3}\right] \)

The outer bound for the capacity region in this regime has a similar shape to the previous regime, but has a different set of corner points. To show that this region is achievable, we are going to show that all the corner points are achievable.
Besides sending a new batch of fresh bits. Transmitter regions. In all time slots, transmitter \( (m(n,m) - \min(\frac{2n-3m}{2}, (l-m)^+) \). The encoding scheme is shown in Figure 8. In the first time slot, transmitter 1 transmits \( (n-m) - \min(\frac{2n-3m}{2}, (l-m)^+) \) fresh bits \( (A_1, A_2) \), and another \( n-m \) fresh bits in the region \( (A_4, A_5) \). No information is encoded in the region \( A_3 \). Transmitter 2 encodes \( (2m-n) \) fresh bits in \( B_1 \), \( E_{211} \) fresh information bits in \( B_2 \), \( E_{212} + E_{213} \) bits in the region \( B_3 \), and \( (2m-n) \) fresh bits in the region \( B_5 \). In the second time slot, besides sending new information as in the first time slot, transmitter 1 relays \( E_{211} \) in the region \( A_2 \). Besides sending new information, transmitter 2 relays \( E_{11} \) in the region \( B_2 \). By repeating those operations again for all other time slots, we can achieve the corner point \( (2(n-m) - \min(\frac{2n-3m}{2}, (l-m)^+), 2(2m-n) + \min(\frac{2n-3m}{2}, (l-m)^+)) \).

C. Moderately Strong Interference: \( \alpha \in [\frac{2}{3}, 1] \)

Case 1: \( l < m \)

In this case, the capacity region is the same as that of the region without any feedback. The encoding schemes for this case are shown in [2].

Case 2: \( m \leq l \leq n \)

The corner points in this case are given by (0, 0), \((0, l), (0, \min(n, l)), (m + (l-m)^+), 2(n-m) - (l-m)^+ \) and \( (2(n-m) - (l-m)^+, m + (l-m)^+) \).

The encoding schemes for the first three points are trivial. In the rest of this subsection, we will show how to achieve the corner point \( (m + (l-m)^+, 2(n-m) - (l-m)^+) \). The scheme to achieve \( (2(n-m) - (l-m)^+, m + (l-m)^+) \) is similar. There are 2 sub-cases to consider here.

Sub-case 2.1: \( 3(n-m) \leq l \)

The encoding scheme for this sub-case is shown in Figure 9. In the first time slot, transmitter 1 transmits \( (2m-n) + (l-m)^+ \) fresh bits in the top region, and \( n-m \) fresh bits in the bottom region. Thus, it sends a total of \( l \) fresh bits. Transmitter 2 transmits \( (n-m) - (l-m)^+ \), or \( n-l \) fresh bits in the top region, and \( n-m \) fresh bits in the bottom region. Thus, it sends a total of \( 2(n-m) - (l-m)^+ \) fresh bits. At the receiving sides, \( E_{11} \) causes interference to receiver 2, and \( F_{21} \) causes interference to receiver 2. Via the feedback link, transmitter 1 can decode \( F_{21} \), and transmitter 1 can decode \( E_{11} \).

In the second time slot, besides sending fresh information as in the first time slot, transmitter 1 relays \( F_{21} \) in the middle gap as shown in the Figure 9, and transmitter 2 relays \( E_{11} \). No matter what value \( l \) takes, receiver 2 can always decode \( E_{11} \), thus can resolve the interference in the first time slot. As a result, it can receive all of \( 2(n-m) - (l-m)^+ \) fresh bits intended for itself in the first time slot. Notice in this sub-case, we have an inequality that always holds: \( n-l \leq 3m-2n \). Thus, \( (n-l) + |E_{11}| \leq (3m-2n) + (l-m) \). Therefore, the bits \( F_{21} \) are always received cleanly in this sub-case. With those bits, receiver 2 can resolve the interference in the first time slot. Those operations are repeated over time. The corner point \( (m + (l-m)^+, 2(n-m) - (l-m)^+) \) is asymptotically achievable.
encoding schemes to achieve the first three corner points are

D. Strong Interference:

Case 1: \( l \leq n \)

The corner points in this case are given by \((0,0), (n,0), (0,n), (n, m - n)\) and \((m - n,n)\). The encoding schemes to achieve the first three corner points are trivial. The encoding schemes to achieve the last two points are the same as the encoding schemes without feedback, and are shown in [2].

Case 2: \( n \leq l \)

The corner points in this case are

Sub-case 2.2: \( l < 3(n - m) \)

The encoding schemes in this case are similar to the first sub-case. Transmitter 1 sends \( l \) fresh information bits, transmitter 2 sends \( 2(n - m) - (l - m) \) fresh bits in the same regions every time slot. From the second time slot onwards, transmitter 2 relays the bits in the region \( E_{l1} \). However, there is a slight variation. Transmitter 1 relays in part, or in whole, the bits in the region \( F_{21} \), which have not been decoded by receiver 1 yet, depending on the relative value of \( l \). The encoding scheme is illustrated by Figure 10.

Using this strategy, receiver 2 can always decode the bits in the region \( E_{l1} \); thus, it can resolve interference caused by transmitter 1 and achieve a rate of \( 2(n - m) - (l - m) \) fresh bits per channel use asymptotically. It can be shown that receiver 1 always achieves a rate of \( l \) bits per channel use.

\[ \frac{3n}{T} \leq m, m \leq l, 3(n - m) \leq l \]

\[ \frac{3m}{T} \leq m, m \leq l, l < 3(n - m) \]

Sub-case 2.2: \( l < 3(n - m) \)

Assume \( l \leq m \); otherwise, the result is trivial. The encoding scheme is shown in Figure 11. In the first time slot, transmitter 1 sends \( l \) fresh bits in the top region. Transmitter 2 sends \( F_{21} \) in the top region, where \( |F_{21}| = 2n - m \). In addition, it sends \( 2m - 2n - l \) bits in the middle region, such that there is a small gap of \( l - n \) bits. Thus, transmitter 2 sends a total of \( m - l \) fresh information bits. At the end of the first time slot, out of \( l \) bits sent from transmitter 1, receiver 1 can receive \( n \) intended bits cleanly. It cannot receive \( E_{l1} \) directly yet. Receiver 2 can receive \( 2m - 2n - l \) bits cleanly and it cannot receive the bits in \( F_{21} \) due to interference. Via the feedback link of strength \( l \) bits, transmitter 2 can decode \( E_{11} \), and transmitter 1 can decode \( F_{21} \). In the second time slot, besides sending new \( m - l \) fresh information, transmitter 1 relays \( F_{21} \) in the bottom region. Besides sending new \( m - l \) fresh information bits, transmitter 2 relays \( F_{21} \) as shown in the diagram. Subsequently, receiver 1 can recover \( E_{11} \), and receiver 2 can recover \( F_{21} \) at the end of the second time slot. Those operations are repeated over time. Asymptotically, the corner point \((l, m - l)\) is achieved.

Sub-case 2.2: \( n \leq l \) and \( 2m - 2n \leq l \)

Assume \( l \leq m \); otherwise, the result is trivial. The encoding scheme is shown in Figure 12. Notice that \( (m - l) + (m - n) \leq n \), which makes the encoding scheme feasible. In the first time slot, transmitter 1 sends \( l \) fresh bits in the top region; transmitter 2 sends \( F_{21} \) in the top region, where \( |F_{21}| = m - l \). At the end of the first time slot, out of \( l \) bits sent from transmitter 1, receiver 1 can receive \( n \) intended bits cleanly. It cannot receive \( E_{l1} \) directly yet. At the end of the first time slot, receiver 2 cannot receive the bits in \( F_{21} \) directly yet as \( F_{21} \) is corrupted by interference from transmitter 1. In the second time slot, besides sending new \( m - l \) fresh information, transmitter 1 relays \( F_{21} \) in the bottom region.
Besides sending new \(m - l\) fresh information bits, transmitter 2 relays \(E_{11}\) as shown in the diagram. Subsequently, receiver 1 can recover \(E_{11}\), and receiver 2 can recover \(F_{11}\) at the end of the second time slot. Operations are repeated over time. Asymptotically, the corner point \((l, m - l)\) is achieved.

E. Very Strong Interference: \(\alpha \in [2, \infty)\)

Case 1: \(2l < m\). The corner points in this case are given by \((0, 0),(n + (l - n)^+, 0),(0, n + (l - n)^+),(l, m - l)\). The achievable scheme for the first three points are trivial. The achievable scheme for the last corner point is given in our ISIT paper [10].

Case 2: \(m < 2l\). The corner points in this case are given by \((0, 0), (0, \min(n, n + (l - n)^+)), (l, m - l)\). We are going to show how to achieve the point \((l, m - l)\). The encoding scheme to achieve this corner point is shown in Figure 13. The set of \(m\) bits at transmitter \(Tx_1\) is partitioned into 3 regions \(A_1, A_2\) and \(A_3\) with respective sizes \(n, m - 2n\) and \(m\). A similar partition is done at transmitter 2.

In the first time slot, transmitter 1 transmits a total of \(l\) fresh bits, \(n\) of which are encoded in \(A_1\), and \((l - n)\) of which are encoding in \((A_2, A_3)\). Note \(|E_{11}| = l - n\). For the second transmitter, there will be two sub-cases. Consider the first sub-case when \(m - l \geq n\). In this sub-case, in the first time slot, \(n\) fresh information bits are encoded in the region \(B_1\), and \(F_{11}\) bits are encoded in the next region as shown in Figure 13. Note \(|F_{11}| = m - l - n\). In the second sub-case, \(m - l < n\), there is a slight difference from the first sub-case. In this sub-case, only \(m - l\) bits are encoded in the region \(B_1\).

Operations in subsequent time slots between these two sub-case are similar; thus we only discuss the first sub-case in detail here. Via the feedback link, transmitter 2 can recover \(E_{11}\) easily. Notice \(m - l < l\), thus transmitter 1 can recover \(F_{11}\) too with a feedback level of \(l\) bits. In the second time slot, besides encoding \(l\) fresh bits, transmitter 1 relays \(F_{11}\) bits in the region \(A_2\). Besides encoding \(m - l\) fresh bits, transmitter 2 relays \(E_{11}\). Due to the very strong interference link, receiver 1 can receive \(E_{11}\) cleanly, and receiver 2 can receive \(F_{11}\) cleanly. Similar operations are done in the subsequent time. By this approach, we can achieve the corner point \((l, m - l)\).

VI. Conclusions

In this paper, we have obtained the capacity region for the symmetric LD-IC with partial feedback. Partial feedback can enlarge the capacity region if the feedback link strength is greater than a certain threshold, which varies for different regimes.

REFERENCES


