Dimensional analysis of the audio signal/noise power in a FM system

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1 Problem statement

Jakes in [1] has presented an analytical result for the audio signal and noise power in a FM receiver under additive white Gaussian noise (AWGN) channel conditions. The audio signal power is given by

\[ S = \frac{\pi^2}{10} (B - 2W)^2, \]

where \( B \) and \( W \) represent the noise bandwidth of the IF filter and the baseband filter cutoff frequency respectively. The audio noise power is given by

\[ N(\rho) = \frac{a(1 - e^{-\rho})^2}{\rho} + \frac{8\pi BW e^{-\rho}}{\sqrt{2(\rho + 2.35)}}, \]

where \( \rho \) is the pre-detection SNR and \( a \) is given by

\[ a = \frac{4\pi^2}{B} \int_0^W f^2 e^{-\pi f^2/B^2} df. \]

We observe that both the audio signal and noise power have dimensions of Hz\(^2\). However, we expect the units of power to be watts (W). The mismatch in the units is probably due to some assumptions in the analytical result. This document details the derivation of the analytical expressions for audio signal and noise power as presented in [1] with some minor modifications to clarify the units.

2 Dimensional analysis of the theoretical result

To analyze the properties of audio noise, we consider a hypothetical FM communication system without any modulating signal. The voltage presented to the FM detector is

\[
e(t) = Q \cos(\omega_c t) + n(t) = [Q + X_c(t)]\cos(\omega_c t) - X_s(t)\sin(\omega_c t) = R(t)\cos[\omega_c t + \theta(t)],
\]

where \( Q \) is the amplitude of the carrier, \( n(t) \) is the additive noise, and \( X_c(t), X_s(t) \) are the in-phase and the quadrature components of the noise \( n(t) \) respectively.

The amplitude, \( R(t) \), and phase, \( \theta(t) \), of voltage \( e(t) \), are given by

\[
R(t) = \sqrt{[Q + X_c(t)]^2 + X_s^2(t)} \quad \text{(7)}
\]

\[
\theta(t) = \arctan \left[ \frac{X_s(t)}{Q + X_c(t)} \right] \quad \text{(8)}
\]
The computation of the output noise is particularly simple when the SNR at IF, \( \rho \), is very large. When \( \rho \gg 1 \), equations (7) and (8) can be approximated as

\[
R(t) \approx Q + X_c(t) \tag{9}
\]
\[
\theta(t) \approx \frac{X_s(t)}{Q} \cdot \tag{10}
\]

The output of the FM differentiator is given by

\[
m_o(t) = \frac{\theta'(t)}{c} \tag{11}
\]

where the prime denotes differentiation with respect to time and \( c \) is the transfer characteristic (gain) of the differentiator. \( c = 2\pi c' \), where \( c' \) is a constant with units of Hz/V. Therefore, from equations (10) and (11),

\[
m_o(t) = \frac{X'_s(t)}{cQ} \tag{12}
\]

The power spectral density of \( X_s(t) \) is given by

\[
X_s(f) = \int_{-\infty}^{\infty} g(\tau) e^{-j2\pi f \tau} d\tau \tag{13}
\]
\[
= \eta [H(f_c - f) + H(f_c + f)] \tag{14}
\]

where \( g(\tau) \) is the autocorrelation of \( X_s(t) \), \( \eta \) is the power spectral density of the AWGN and \( H(f) \) denotes the frequency response of the IF filter centered at \( f_c \). From equation (14), we see that the units of \( P_{X_s}(f) \) is V\(^2\)/Hz, same as that of \( \eta \).

Since the power spectrum of the derivative of a stationary stochastic process is \((2\pi f)^2\) times the spectrum of the process itself [2], we have from equations (12) and (14) the PSD of \( m_o(t) \) is

\[
P_{m_o}(f) = \left(\frac{2\pi f}{cQ}\right)^2 \eta [G(f_c - f) + G(f_c + f)] \tag{15}
\]

So we see that from equation (15), the units of \( P_{m_o}(f) \) is V\(^2\)/Hz.

Rice in [3] divided the one-sided baseband noise spectrum, \( P(f) \), into three components:

\[
P(f) = P_1(f) + P_2(f) + P_3(f) \cdot \tag{16}
\]

\( P_1(f) \) has the same spectrum shape as the output noise spectrum when the carrier is absent. \( P_2(f) \) has the shape of the output noise spectrum when the carrier is very large,

\[
P_2(f) = (1 - e^{-\rho})^2 P_{m_o}(f) \tag{17}
\]

where \( P_{m_o}(f) \) is the output noise spectrum when \( \rho \gg 1 \), given in equation (15). \( P_3(f) \) is a correction term that predominates in the threshold region of \( \rho \). Davis in [4] has shown that in the
frequency range from 0 to $W$, where $W < B$, the spectral components $P_1(f) + P_3(f)$ may be accurately approximated by

$$P_D(f) \approx 8\pi B e^{-\rho} \left[ c^4 \sqrt{2(\rho + 2.35)} \right]^{-1/2}.$$  \hfill (18)

The above approximations may be combined to provide an approximation for the overall baseband noise spectrum:

$$P(f) \approx (1 - e^{-\rho})^2 P_{m_o}(f) + P_D(f) \approx \frac{[2\pi f (1 - e^{-\rho})] e^{-\pi f^2/B^2}}{c^2 B \rho} + \frac{8\pi B e^{-\rho}}{c^2 \sqrt{2(\rho + 2.35)}},$$  \hfill (19)

where we have used a Gaussian IF filter shape given by

$$H(f) = e^{-\pi(f-f_c)^2/B^2}.$$  \hfill (20)

Since the baseband filter width is usually narrow relative to the IF bandwidth ($W < B$), the exact shape of the IF filter has little effect on the output noise. Thus, we lose little generality by including the assumption of a Gaussian shape for the IF filter.

The components $P_2(f)$ and $P_D(f)$, both have dimensions of V$^2$/Hz. The total noise out of a rectangular baseband filter is then

$$N(\rho) = \int_0^W P(f) df$$  \hfill (21)

So the audio noise power, $N(\rho)$, has dimensions of W, assuming unit impedance.

Now let us derive the expression for output signal power and perform a dimensional consistency check on it. As the IF SNR falls and goes below the threshold, the signal modulation is suppressed. So the signal output from the discriminator is given by \[4, 5\]

$$m_o(t) = m(t)(1 - e^{-\rho}).$$  \hfill (22)

Hence the output baseband signal power, $S_0$, is related to the input modulation signal power, $S = \langle m^2(t) \rangle$, by

$$S_0 = (1 - e^{-\rho})^2 S.$$  \hfill (23)

Here the angle brackets $\langle . \rangle$ denotes time averaging. Let $\sigma_m$ denote the rms frequency deviation of the signal $c^\prime m(t)$. Hence its power is $\sigma_m^2$. Then the IF bandwidth used is chosen according to the formula $B = 2(W + \sigma_m \sqrt{10})$, which ensures that modulation peaks less than $\sigma_m + 10$ dB are contained within the deviation given by Carson’s rule for sinusoidal modulation. Thus we have \[4\],

$$S = \frac{\sigma_m^2}{c^2},$$  \hfill (24)

$$= \frac{4\pi^2 \sigma_m^2}{c^2},$$  \hfill (25)

$$= \frac{\pi^2 (B - 2W)^2}{10c^2}.$$  \hfill (26)
From equations (23) and (24), and assuming unit impedance, we find that the dimensions of output signal power, $S_o$, is also W.

3 Result

This document presented the original Jakes result \cite{1} for the audio signal and noise power in FM receiver under AWGN conditions. We observed that the original expression for both the signal and noise power have dimensions of Hz$^2$. We then introduced explicitly the transfer characteristic gain of the discriminator, $1/c$ [V/Hz]. The units of the modified audio signal and noise power expressions are determined to be consistent, and the resulting signal to noise power is independent of this constant.

References


